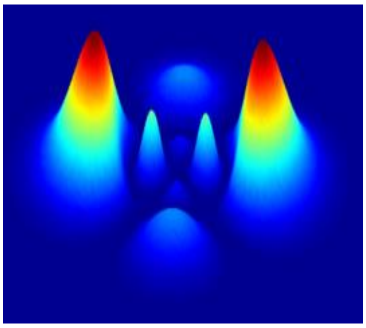


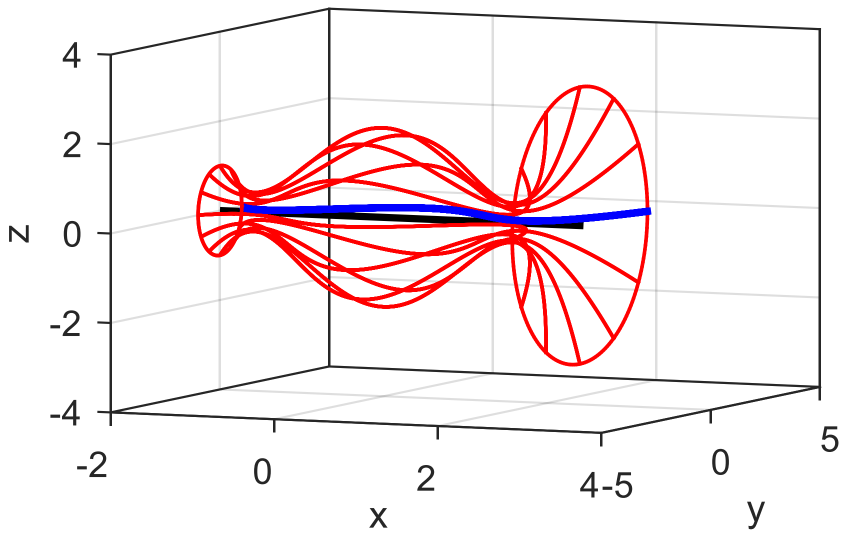
ADVANCED HIGH SCHOOL MATHEMATICS

VOLUMES

DISK METHOD

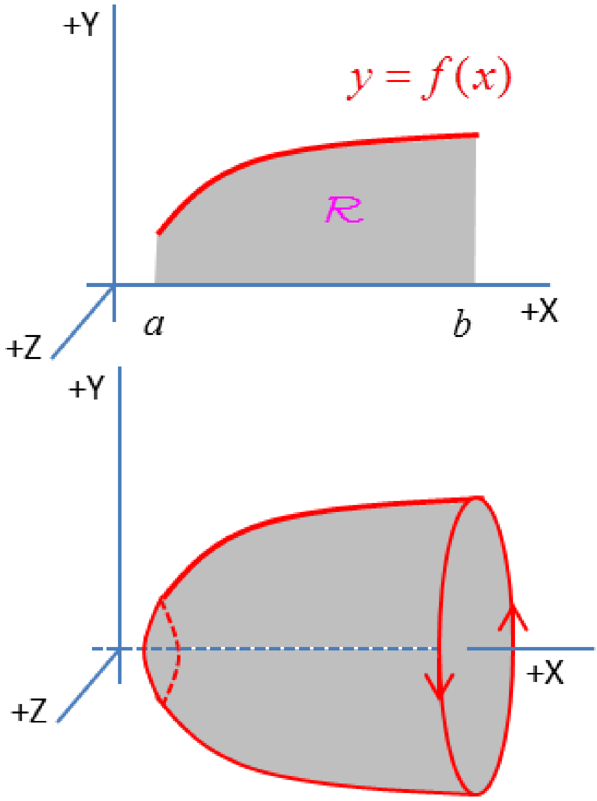
Ian Cooper email: matlabvisualphysice@gmail.com

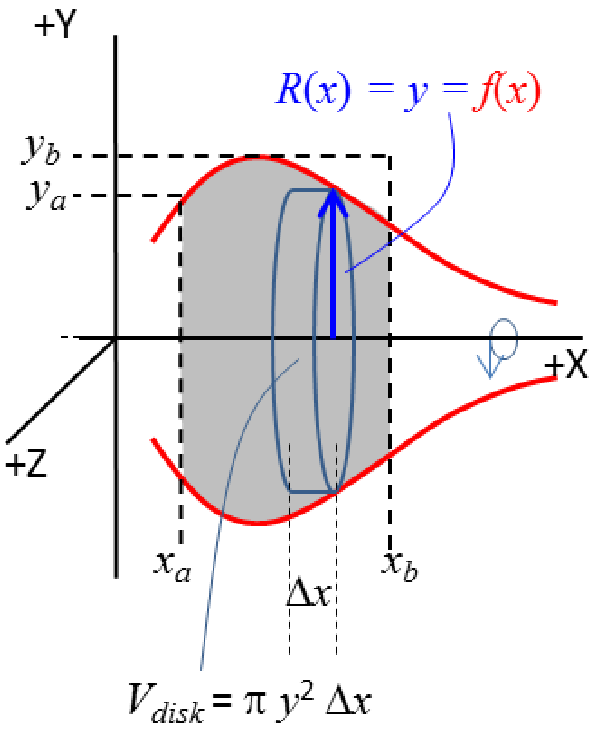
A **solid of revolution** is obtained by revolving a **region** in a plane about a straight line called the **axis of revolution** that does not intersect the region



DISK METHOD: ROTATION ABOUT THE X AXIS

Let $f(x)$ be a continuous single valued function such that $f(x) \geq 0 \quad x_a = a \leq x \leq x_b = b$. The region $\mathcal{R}$ is the area bounded by the function $y = f(x)$ and the X-axis in the interval $[a, b]$. Revolving of the region $\mathcal{R}$ about the X-axis generates a solid called a **solid of revolution**.





$y = f(x)$ is a single valued function and the region $\mathcal{R}$ is bounded by the function $y = f(x)$, the X-axis and the vertical lines x_a and x_b. $R(x)$ is the radius function.
The Y-axis limits are $y_a = f(x_a)$ $\quad y_b = f(x_b)$

The volume V of the solid of revolution obtained by revolving a region $\mathcal{R}$ about the X-axis is

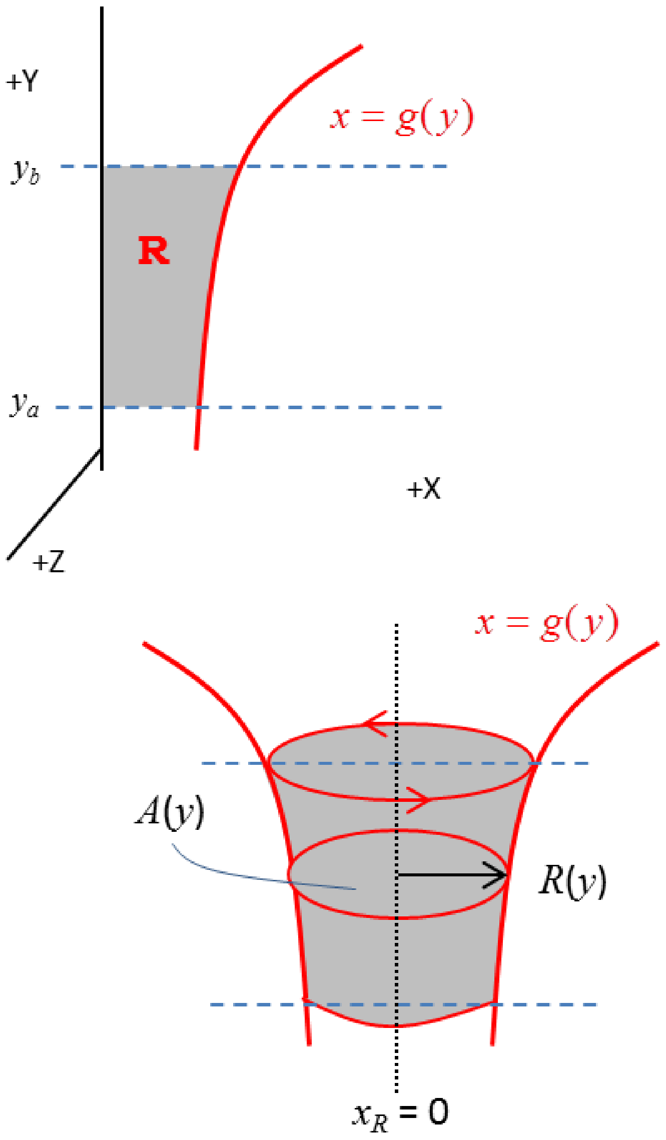
$$V = \int_{x_a}^{x_b} A(x)\,dx$$

$$A(x) = \pi\, f(x)^2 = \pi\, R(x)^2 = \pi\, y^2$$

$$V = \pi \int_{x_a}^{x_b} f(x)^2\,dx = \pi \int_{x_a}^{x_b} R(x)^2\,dx = \pi \int_{x_a}^{x_b} y^2\,dx$$

DISK METHOD: ROTATION ABOUT THE Y AXIS

Let $x = g(y)$ be a single-valued continuous function where $g(y) \geq 0$ in the interval $y_a \leq y \leq y_b$. Consider the region $\mathcal{R}$ bounded by the function $x = g(y)$ and the Y-axis ($x_R = 0$) for the interval $y_a \leq y \leq y_b$.



When this region $\mathcal{R}$ is rotated about Y-axis through the 360° rotation, a solid of revolution is generated.

The volume V of the solid of revolution is given by

$$V = \int_{y_a}^{y_b} A(y)\,dy$$

The solid generated by the rotation must have a circular cross-section with radius $R(y)$. Therefore, the cross-sectional area $A(y)$ is given by

$$A(y) = \pi\, R(y)^2 \quad R(y) = x \quad A(x) = \pi\, x^2$$

$$V = \pi \int_{y_a}^{y_b} R(y)^2\, dy = \pi \int_{y_a}^{y_b} x^2\, dy$$

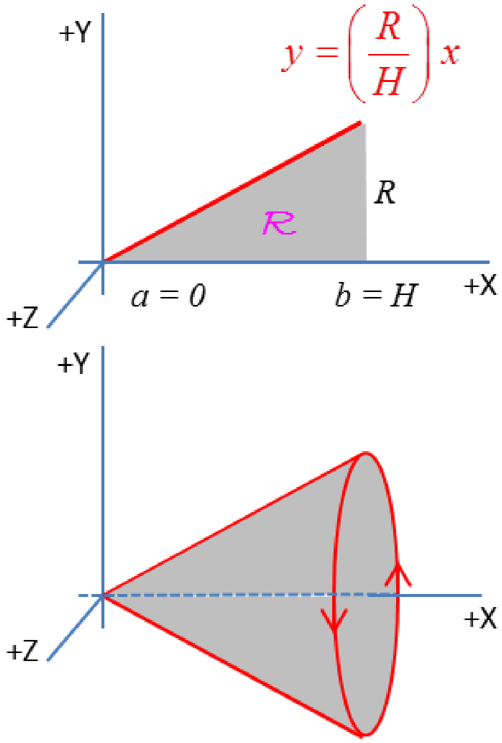
Example Pyramid with circular base

Find the volume V of a cone of height H and base radius R,

generated by revolving the function $y = \left(\frac{R}{h}\right)x$ about the X-axis.

Solution

How to approach the problem: Draw the XYZ axes / Sketch the function and the solid / Give the equation for the shape of the solid / Evaluation the definite integral to find the volume.



Volume of solid of revolution about the X axis is

$$V = \pi \int_a^b y^2 \, dx$$

The limits of integration are $a = 0$ and $b = H$.

The volume of the cone is

$$V = \pi \int_0^H \left(\frac{R}{H} \right)^2 x^2 \, dx = \pi \left(\frac{R}{H} \right)^2 \left[\tfrac{1}{3} x^3 \right]_0^H$$

$$V = \tfrac{1}{3} \pi R^2 H$$

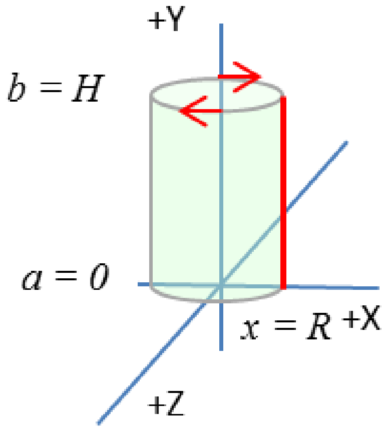
QED

Example Cylinder

Find the volume V of a cylinder of height H and radius R by evaluation of a definite integral.

How to approach the problem: Draw the XYZ axes / Sketch the function and the solid / Give the equation for the shape of the solid / Evaluation the definite integral to find the volume.

Solution



Rotate the line $x = R$ about the Y-Axis to generate the cylinder.

Volume of solid of revolution about the Y-axis is

$$V = \pi \int_a^b x^2 \, dy$$

The limits of integration are $a = 0$ and $b = H$.

The volume V of the cylinder is

$$V = \pi \int_0^H R^2 \, dy = \pi R^2 \left[y \right]_0^H$$

$$V = \pi R^2 H$$

QED

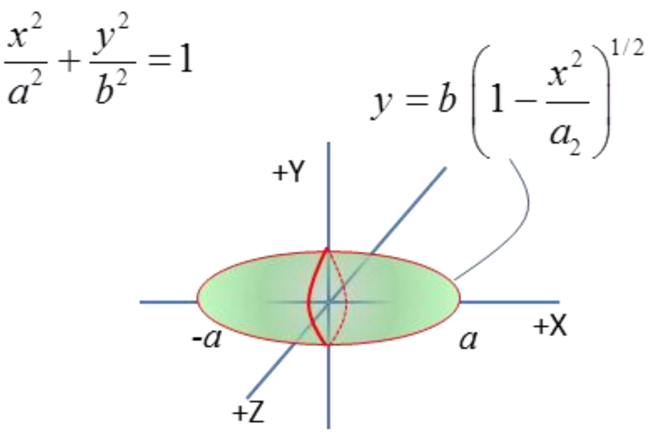
Example Ellipsoid

Find the volume V of an ellipsoid formed by the rotation of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$ about the X-axis.

Solution

How to approach the problem: Draw the XYZ axes / Sketch the function and the solid / Give the equation for the shape of the solid / Evaluation the definite integral to find the volume.



Volume of solid of revolution about the X axis is

$$V=\pi\int_a^b y^2\,dx$$

The limits of integration are $x_a = -a$ and $x_b = a$.

The function $y = f(x) \geq 0$ in the interval [- a a] is

$$y = b\left(1 - \frac{x^2}{a_2}\right)^{1/2} \qquad y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

The volume of the ellipsoid is

$$V = \pi \int_{-a}^{a} b^2 \left(1 - \frac{x^2}{a^2}\right) dx = \frac{2\pi b^2}{a^2} \int_0^a \left(a^2 - x^2\right) dx$$

$$V = \frac{2\pi b^2}{a^2}\left[a^2 x - \tfrac{1}{3} x^3\right]_0^a$$

$$V = \frac{4\pi a b^2}{3}$$

For a sphere of radius a $\quad a = b \quad V_{sphere} = \dfrac{4\pi a^3}{3}$

QED

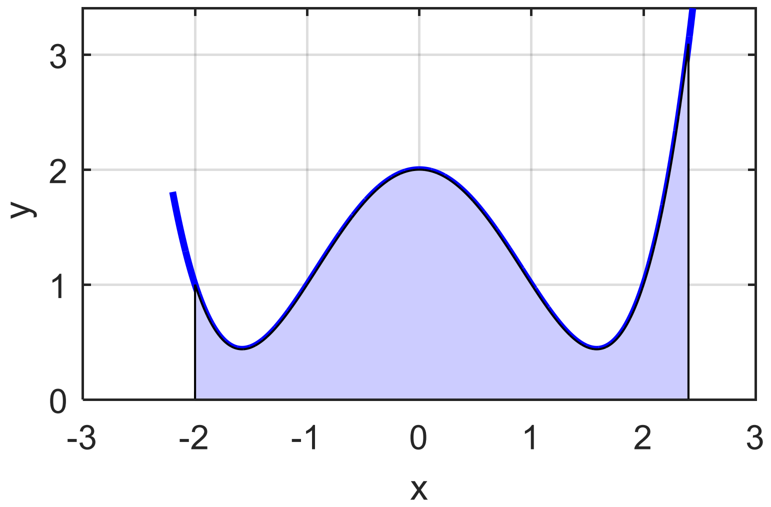
Example

The region $\mathcal{R}$ is the blue shaded area under the curve which is rotated about the X-axis. The function

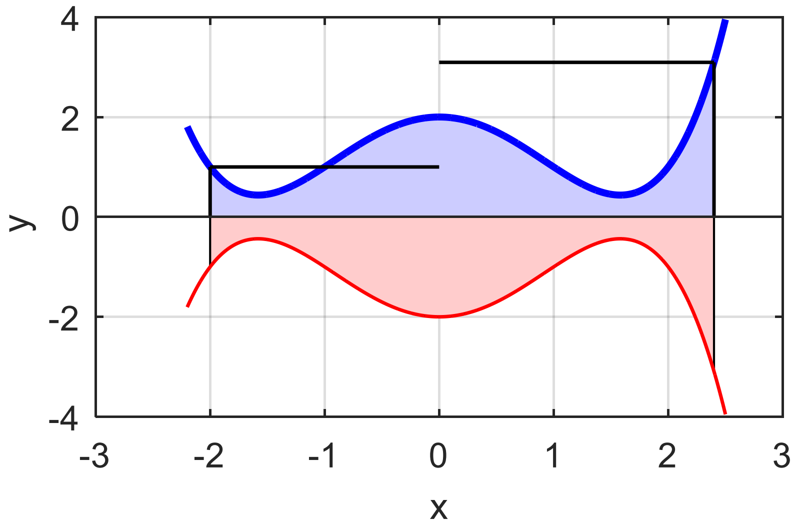
$$y = f(x) = 2 + \left(x^4 - x^2\right)/4$$

is shown by the blue curved line. The region is bounded by the X-axis and the lines x_a = -2 and x_b = +2.4.

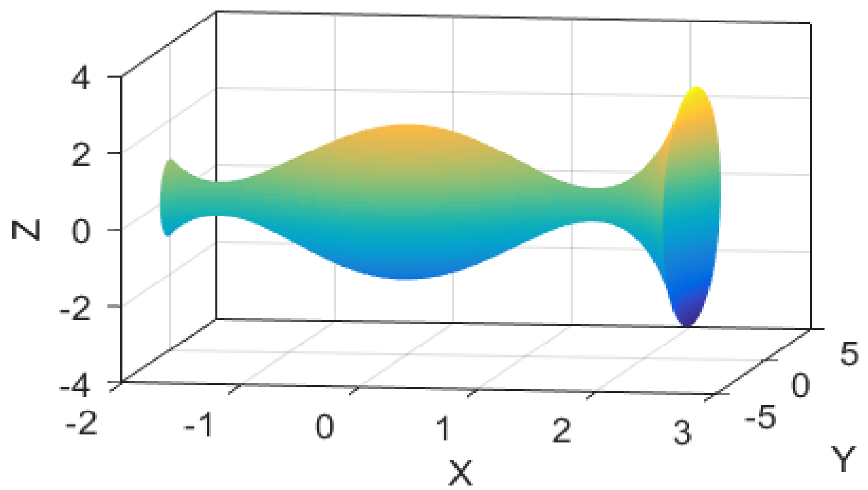
The Y limits are $y_a = +1$ and $y_b = 3$.



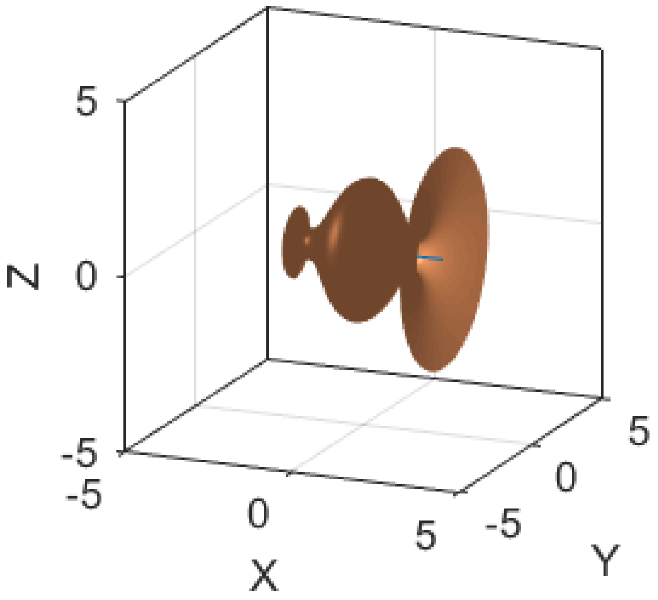
The blue shaded region is rotated around the X-axis. The red shaded area shows the reflection of the blue shaded area about the X-axis. The blue and red lines give the profile of the solid of revolution in the XY plane.



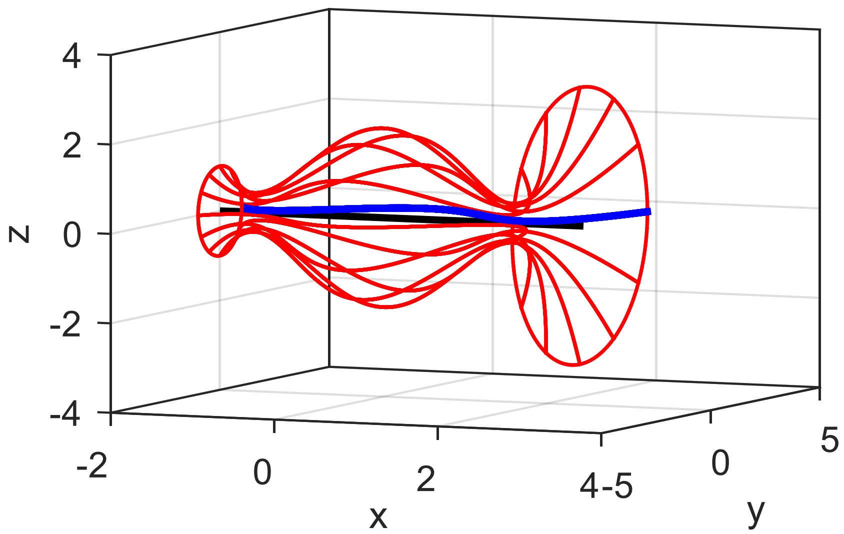
[3D] plot of the solid of revolution



[3D] plot of the solid of revolution (x, y and z values too scale)



[View animation of the rotation of the function]

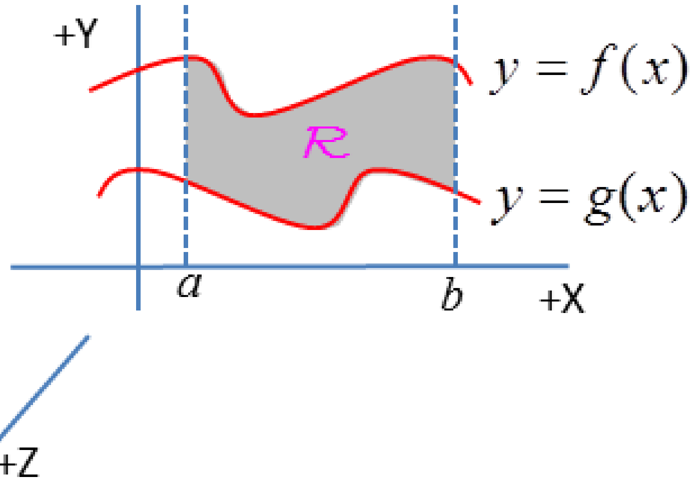


RING OR ANNULUS METHOD

Let $f(x)$ and $g(x)$ be continuous functions such that

$$0 \le g(x) \le f(x) \quad a \le x \le b\,.$$

The region $\mathcal{R}$ is the area bounded by the function $y = f(x)$ and $y = g(x)$ in the interval [*a*, *b*]. Revolving of the region $\mathcal{R}$ about the X-axis generates a solid called a **solid of revolution**.



The solid of revolution has a volume V which is the difference between the volume of revolution generated by the region under $y = f(x)$ and the volume of the solid of revolution generated by the region under $y = g(x)$. Hence, the volume V of the solid of revolution about the X-axis is given by

$$V = \pi \int_a^b \left[\left(f(x) \right)^2 - \left(g(x)^2 \right) \right] dx$$

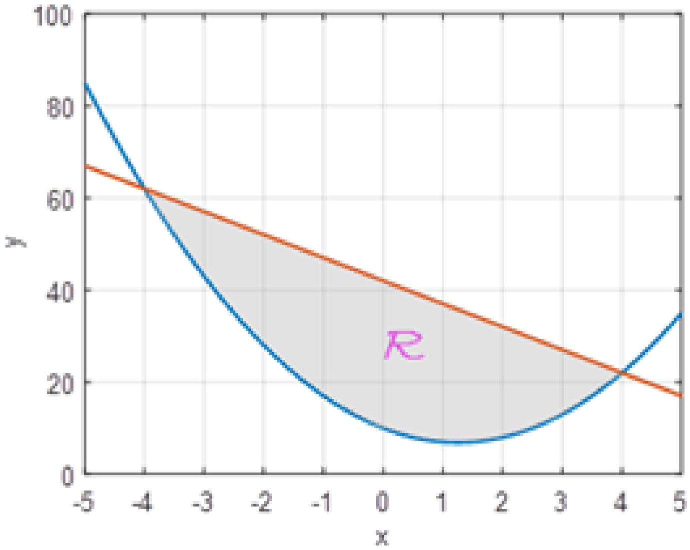
ring / annulus / washer formula[*]

* washer cross section obtained by revolving a vertical segment has the shape of a plumbers washer.

Example

Find the volume V of the solid of revolution generated by the rotation about the X-axis of the region bounded by the curves

$$f(x) = 42 - 5x \quad \text{and} \quad g(x) = 2x^2 - 5x + 10$$



$f(x) = 42 - 5x$

$g(x) = 2x^2 - 5x + 10$

Solution

How to approach the problem:

Sketch the function and the solid.

Give the equations for the shape of the solid. Find the upper and lower limits for the bounded region.

Evaluation the definite integral to find the volume.

Volume of solid of revolution about the X-axis is

$$V = \pi \int_a^b \left[\left(f\left(x\right) \right)^2 - \left(g\left(x\right)^2 \right) \right] dx$$

$$f(x) = 42 - 5x \qquad \left(f(x) \right)^2 = 25x^2 - 420x + 1764$$

$$g(x) = 2x^2 - 5x + 10$$
$$\left(g(x) \right)^2 = 4x^4 - 20x^3 + 65x^2 - 100x + 100$$

$$\left(f(x) \right)^2 - \left(g(x) \right)^2 = -4x^4 + 20x^3 - 45x^2 - 320x + 1664$$

We need to find the points of intersection of the two functions.

$$f(x) = g(x)$$
$$2x^2 - 5x + 10 = 42 - 5x$$
$$x^2 = 16$$
$$x = \pm 4$$

The limits of integration are a = - 4 and b = 4.

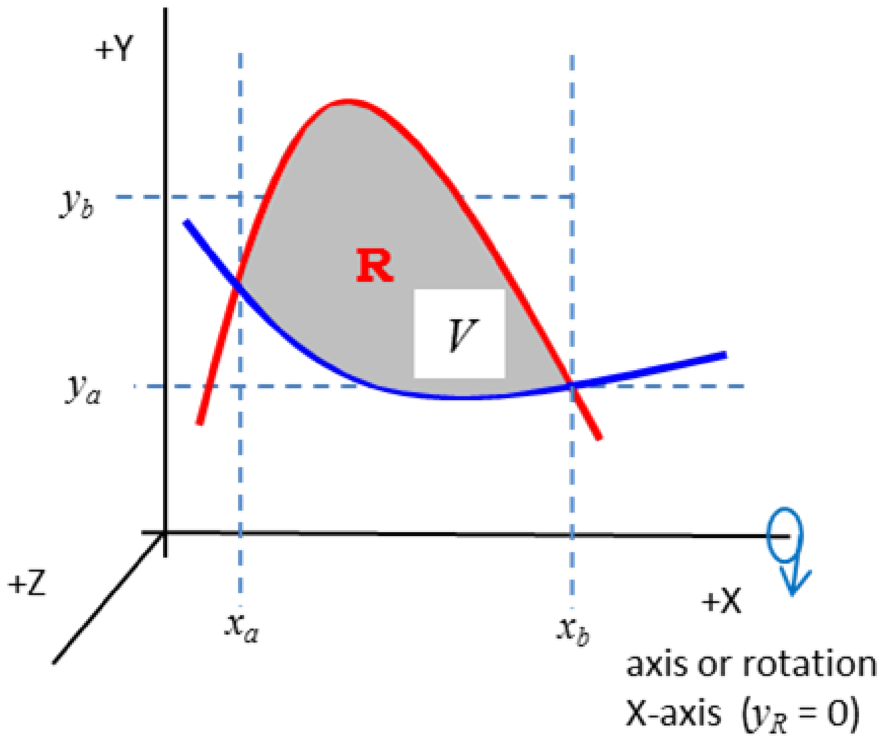
The volume is

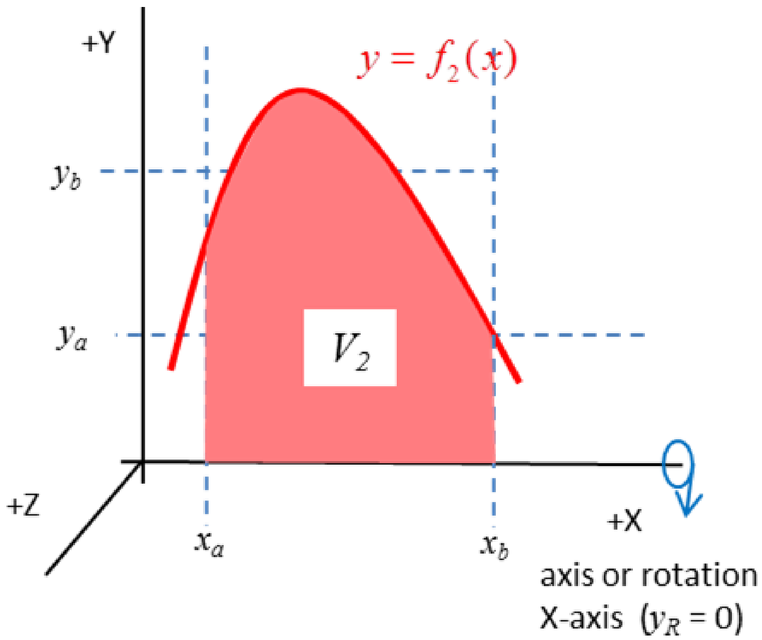
$$V = \pi \int_{-4}^{4} \left[-4x^4 + 20x^3 - 45x^2 - 320x + 1664 \right] dx$$
$$V = \pi \left[-\tfrac{4}{5}x^5 + 5x^4 - 15x^3 - 160x^2 + 1664x \right]_{-4}^{4}$$
$$V = 9754\pi$$

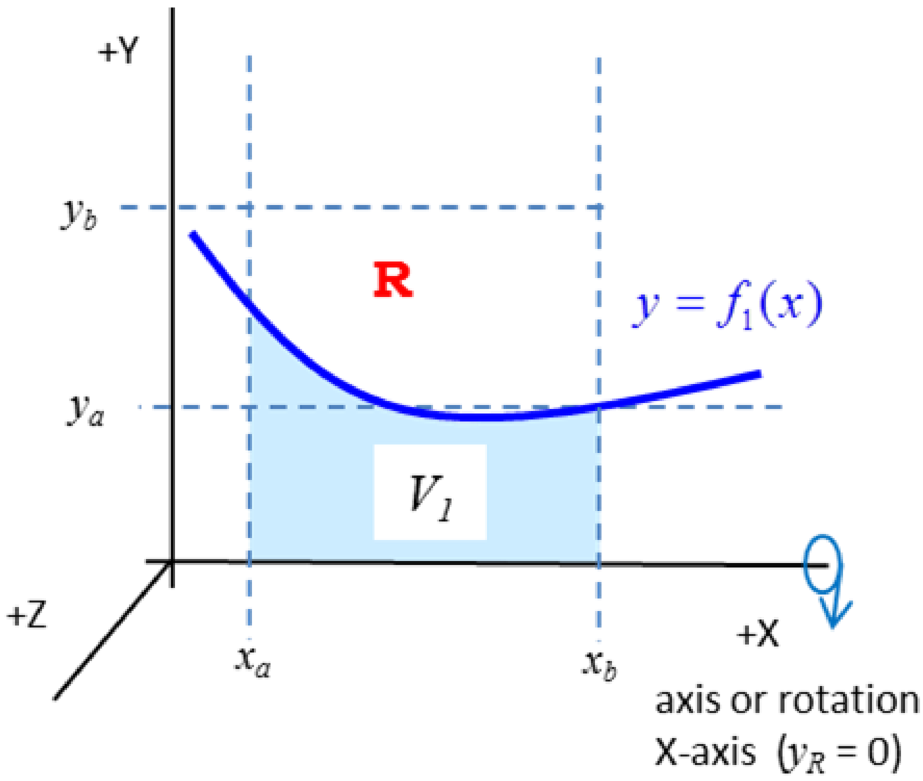
QED

Volumes generated by the rotation of bounded regions

When the region $\mathcal{R}$ between two functions defined in the XY plane is rotated, a solid of revolution can be produced. Let $y = f_1(x)$ and $y = f_2(x)$ be two functions such that $0 \le f_1(x) \le f_2(x)$ in the interval $x_a \le x \le x_b$. The volume V of revolution of the region $\mathcal{R}$ is the difference between the volume V_1 of revolution of $f_1(x)$ and volume V_2 of revolution of $f_2(x)$.





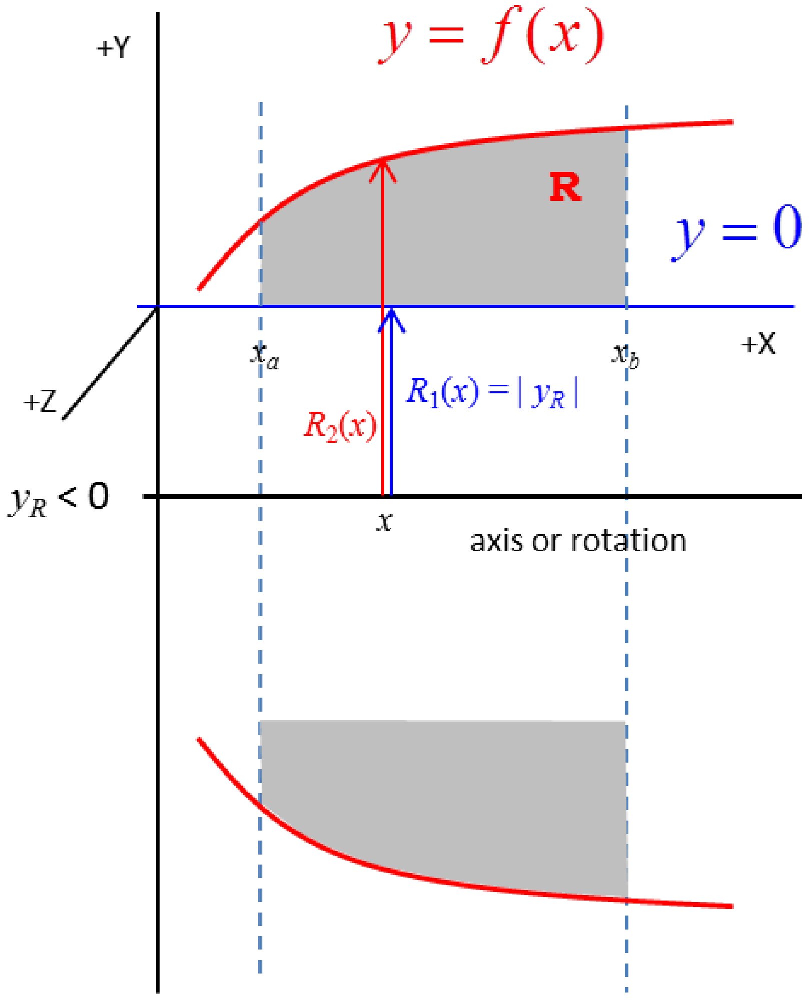


$$V = V_2 - V_1$$

$$V_1 = \pi \int_{x_a}^{x_b} f_1(x)^2 \, dx \qquad V_2 = \pi \int_{x_a}^{x_b} f_2(x)^2 \, dx$$

Solids of revolutions about lines parallel to a coordinate axis

We can also find the volume of revolution when the region is revolved about a line parallel to a coordinate axis. To do such calculations it is necessary to draw a "good quality" sketch of the region to be rotated and the axis of rotation y_R = constant.



The volume V of the solid of revolution for the region R bounded by $y = f(x)$, the X-axis ($y = 0$) and the vertical lines x_a and x_b about the line y_R ($y_R < 0$) is

$$V = V_2 - V_1$$

$$V_1 = \pi \int_{x_a}^{x_b} R_1(x)^2 \, dx \qquad V_2 = \pi \int_{x_a}^{x_b} R_2(x)^2 \, dx$$

where

$$R_2(x) = f(x) - y_R = f(x) + |y_R|$$
$$R_1(x) = -y_R = |y_R| \quad y_R < 0$$

For the rotation about an axis of rotation when

$$y_R \ge f(x) \quad x_a \le x \le x_b$$

$$R_2(x) = y_R \quad y_R \ge f(x)$$
$$R_1(x) = f(x)$$

For rotations about lines parallel to the Y-axis, then the x and y values are simply interchanged.