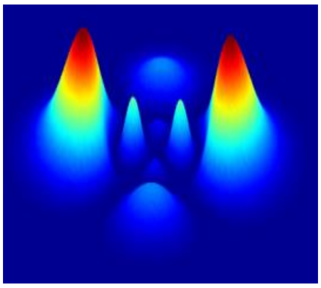


ADVANCED HIGH SCHOOL MATHEMATICS

VOLUMES

CYLINDRICAL SHELL METHOD

In the disk method we integrate along the coordinate axis parallel to the axis of revolution, whereas in the method of cylindrical shells, we integrate along the coordinate axis perpendicular to the axis of revolution. The ability to choose which variable of integration we want to use can be a significant advantage with more complicated functions. Also, the specific geometry of the solid sometimes makes the method of using cylindrical shells more appealing than using the disk method.

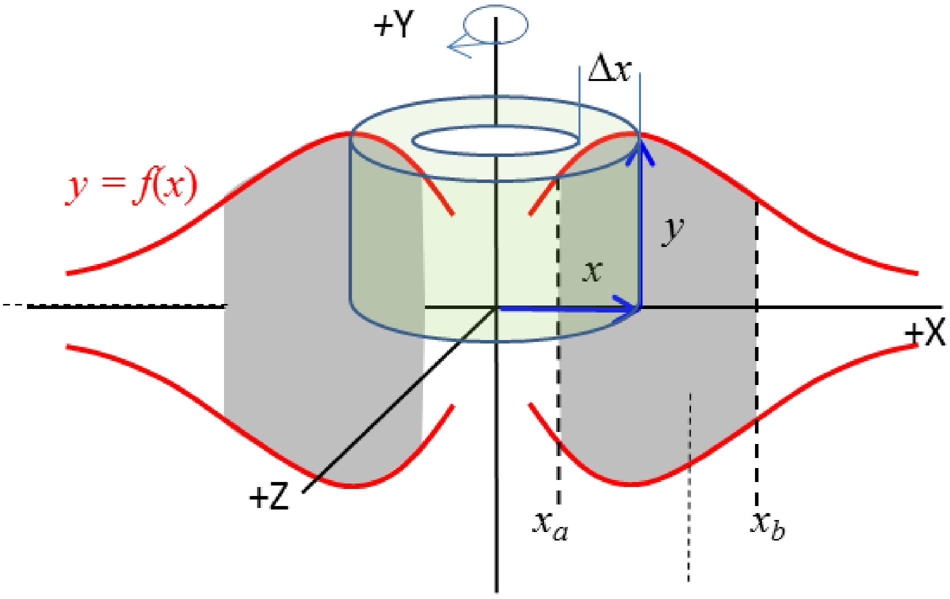
In the cylindrical shell method, to find the volume *V* of a solid, we decompose the solid into hollow concentric shells or rings which are infinitesimally thin and consider rotations about the X or Y axes.

This is often the best method to use for the rotation around the Y-axis of the region bounded by the X-axis and a function $y = f(x)$.

ROTATION ABOUT THE Y-AXIS

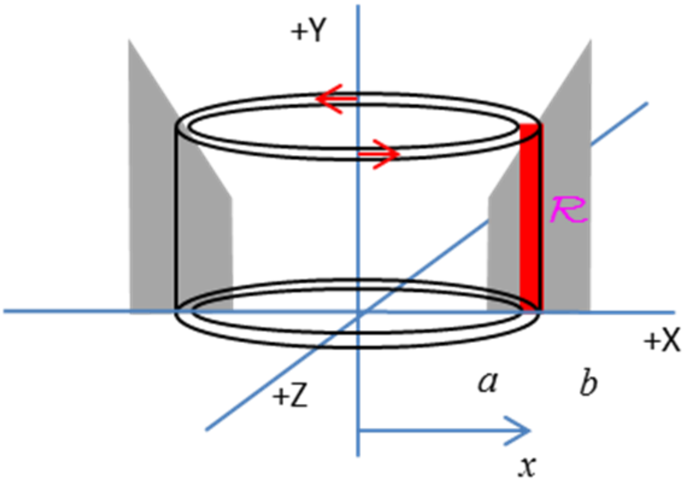
$y = f(x)$ is a single valued continuous function and the region $\mathcal{R}$ is bounded by the function $y = f(x)$, the X-axis and the vertical lines x_a and x_b. The Y-axis limits are $y_a = f(x_a)$ $y_b = f(x_b)$

$$f(x) \geq 0 \quad a \leq x \leq b \quad a \geq 0$$



$$V_{shell} = (2\pi x)\,\Delta x\; y$$

$$V = 2\pi \int_{x_a}^{x_b} y\, x\, dx$$



The volume V of the solid of revolution obtained by revolving a region $\mathcal{R}$ about the Y-axis is derived as follows.

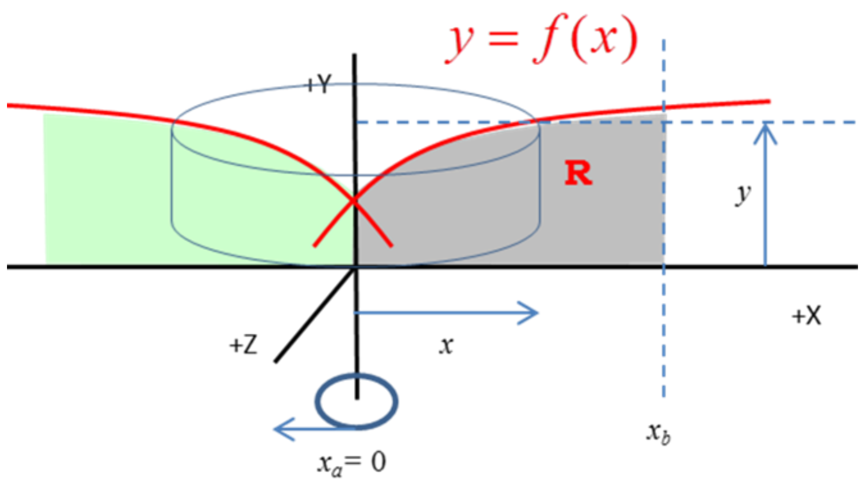
Consider a thin strip of the region $\mathcal{R}$ at position x and of width Δx and height $y = f(x)$. When this strip is rotated around the Y-axis it sweeps out a cylindrical shell of radius x, thickness Δx and height $y = f(x)$.

The volume of this thin cylindrical shell is

$$V_{shell} = \left(2\pi\, x\, \Delta x\right) y$$

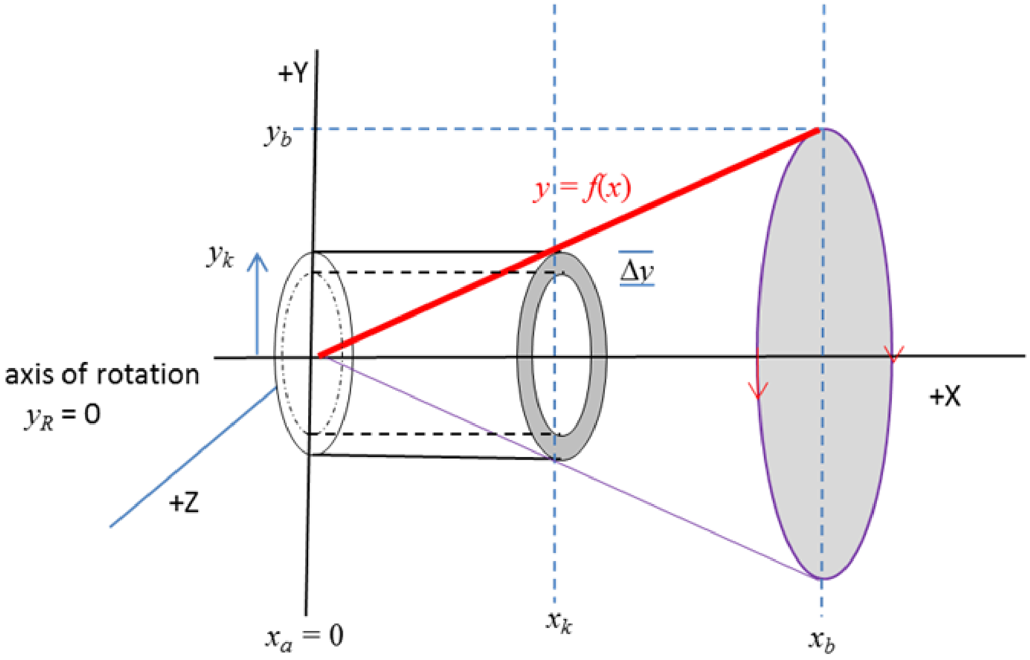
By letting $\Delta x \to 0$ and summing the contribution of each shell for x in the interval $a \le x \le b$, the volume V of the solid of revolution is given by the definite integral

$$V = 2\pi \int_{x_a}^{x_b} y\, x\, dx$$



ROTATION ABOUT X AXIS

In another approach to find the volume V, we decompose the solid into hollow concentric shells or rings which are infinitesimally thin. For a rotation about the X-axis ($y_R = 0$)



The area of each ring is $A(x_k) = 2\pi y_k \, \Delta y$ (circumference x width)

The volume of each the cylindrical shell is

$$V(x_k) = (2\pi \, y \, \Delta y) \, x_k \qquad \text{(area x height)}$$

The total volume V of the solid of revolution is

$$V = \sum_{k=1}^{N} V(x_k) = \sum_{k=1}^{N} (2\pi \, x_k \; y \,) \Delta y$$

As the thickness of the shells becomes infinitely thin

$$N \to \infty \quad \Delta y \to 0 \quad \sum \ \to \int$$

$$V = 2\pi \int_0^{y_b} x\, y\, dy$$

$$V = 2\pi \int_0^{y_b} g(y)\, y\, dy$$

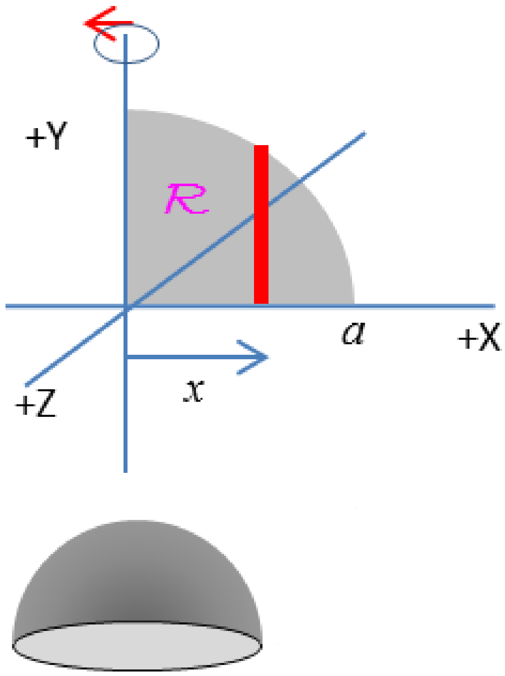
LibreTexts Mathematics

Example

Consider the region $\mathcal{R}$ formed by the part of a circle of radius a in the first quadrant and the X-axis. Revolution about the Y-axis of the region $\mathcal{R}$ produces a solid hemisphere. Find the volume V of the hemisphere by the cylindrical shell method and the disk method.

Solution

How to approach the problem / Sketch the function and the solid / Give the equations for the shape of the solid /Find the upper and lower limits for the bounded region / Evaluation the definite integral to find the volume.



Volume of solid of revolution about the Y-axis using the **cylindrical shell method** is

$$V = 2\pi \int_a^b x\, y\, dx$$

The equation of the circle is $x^2 + y^2 = a^2$

For the region of the circle in the first quadrant

$$y = \left(a^2 - x^2\right)^{1/2}$$

and the limits of integration are 'a' = 0 and 'b' = 0

The volume is

$$V = 2\pi \int_0^a x\, y\, dx = \int_0^a x\left(a^2 - x^2\right)^{1/2} dx$$

$$u = a^2 - x^2 \quad du = -2\, x\, dx \quad x\, dx = \left(-1/2\right) du$$

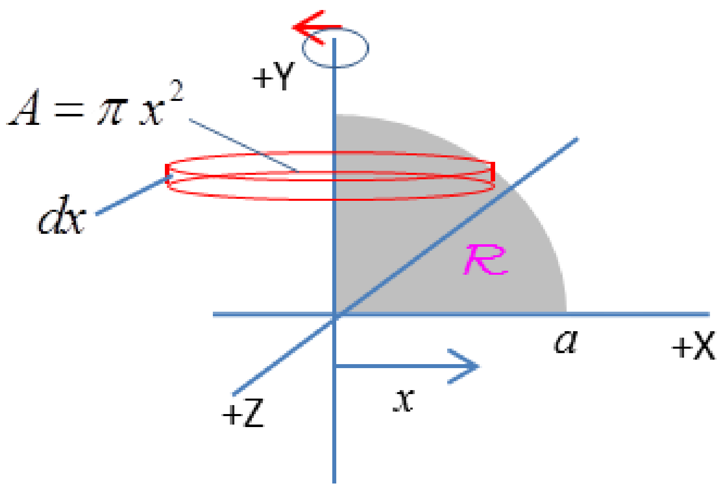
$$x = 0 \quad u = a^2 \qquad x = a \quad u = 0$$

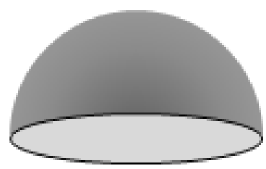
$$V = -\pi \int_{a^2}^0 u^{1/2} du = \pi \int_0^{a^2} u^{1/2} du$$

$$V = \left(\frac{2\pi}{3}\right) a^3$$

QED

Disk Method





Volume of solid of revolution about the Y-axis using the **disk method** is

$$V = \pi \int_a^b x^2 \, dy$$

The equation of the circle is $x^2 + y^2 = a^2$

For the region of the circle in the first quadrant

$$x^2 = \left(a^2 - y^2\right)$$

and the limits of integration are ‘a’ = 0 and ‘b’ = 0

The volume is

$$V = \pi \int_0^a \left(a^2 - y^2\right)^2 \, dy$$

$$V = \pi \left[a^2 y - \tfrac{1}{3} y^3\right]_0^a = \pi \left(a^3 - \tfrac{1}{3} a^3\right)$$

$$V = \left(\frac{2\pi}{3}\right) a^3$$

QED