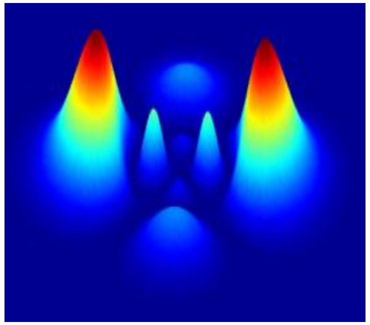

ADVANCED HIGH SCHOOL MATHEMATICS

VOLUMES

VOLUMES BASED UPON CROSS SECTIONS

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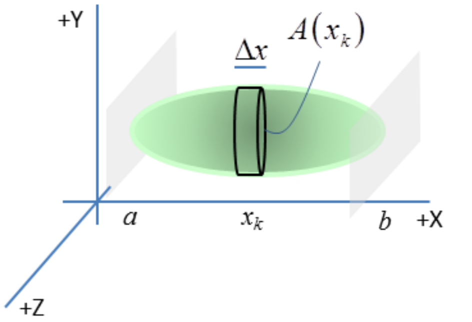
We can calculate the volume of a solid by dividing it into N small volume elements and use a procedure similar to finding the area under a curve. The k^{th} element has a volume $A(x_k)\Delta x$ where Δx is the width of the element and $A(x_k)$ is the cross-sectional area of the solid at position x_k. Assume that the solid (not necessarily a solid of revolution) lies entirely between the plane perpendicular to the X-axis at $x = a$ and the plane perpendicular to the X-axis at $x = b$. The areas $A(x_k)$ all lie in a plane perpendicular to the X-axis. The approximate volume V of the solid is

$$V \approx \sum_{k=1}^{N} A(x_k)\Delta x$$



As $\Delta x \to 0$ we get a better approximation and we can replace the summation by the definite integral

$$V = \int_a^b A(x)\,dx$$

Example

Assume a solid of length L is such that a cross-section perpendicular to the axis of the solid at a distance x from the end at O is a circle of radius $\sqrt{k\,x}$.

Find the volume of the solid.

How to approach the problem:
Draw the XYZ axes.
Sketch the solid aligned along the X-axis.
Give the equation for the shape of the solid.
Express the cross-sectional area A as a function of x.
Evaluation the definite integral to find the volume.

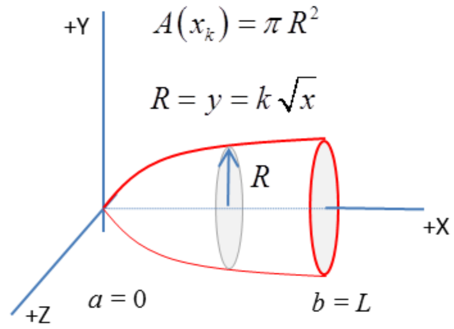
Solution

The radius of the solid is given by $R(x) = y = k\sqrt{x}$

The cross-sectional area is $A(x) = \pi R^2 = k^2\,x$

The limits of the integration are $a = 0 \quad b = L$

The volume of the solid is given by the definite integral



$$V = \int_a^b A(x)\,dx = \int_0^L \pi\,k^2\,x\,dx$$

$$V = \tfrac{1}{2}\pi\,k^2\,L^2$$

QED

Example Pyramid with square base

Find the volume of a pyramid of height H with a square base with sides of length a.

Solution

How to approach the problem:
Draw the XYZ axes.
Sketch the solid aligned along the X-axis.
Express the cross-sectional area A as a function of x.
Evaluation the definite integral to find the volume.

The volume of the solid is given by the definite integral

$$V = \int_a^b A(x)\,dx$$

The cross-sections are squares and the area $A(x)$ is

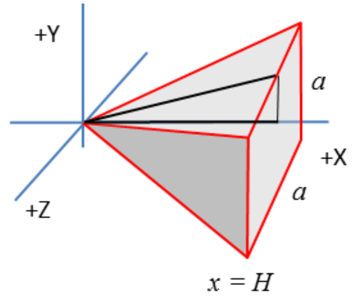
$$A(x) = \left(\frac{a^2}{H^2}\right)x^2$$

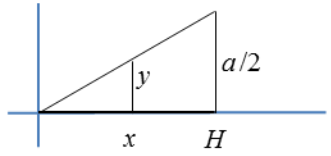
The limits of the integration are $a = 0 \quad b = H$

$$V = \int_a^b A(x)\,dx = \int_0^H \left(\frac{a^2}{H^2}\right)x^2\,dx$$

$$V = \tfrac{1}{3}a^2 H$$

QED





$$\frac{y}{x} = \frac{a}{2H} \qquad y = \left(\frac{a}{2H}\right)x$$

$$A(x) = (2y)^2 = \left(\frac{a^2}{H^2}\right)x^2$$