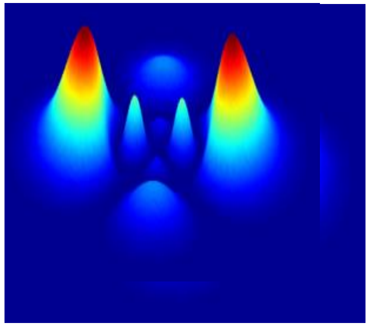

ADVANCED HIGH SCHOOL MATHEMATICS

POLYNOMIALS

FUNDAMENTAL THEOREM OF ALGEBRA

COMPLEX ROOTS AND MULTIPLE ROOTS

Fundamental Theorem of algebra: every polynomial can be factored (over the real numbers) into a product of linear factors and irreducible quadratic factors.

The Fundamental Theorem of Algebra was first proved by **Carl Friedrich Gauss** (1777-1855).

Whenever a polynomial has been factored into only **linear** and **irreducible quadratics**, then it has been **factored completely**, since both linear factors and irreducible quadratics cannot be factored any further over the real numbers.

There are **no** general rules for factoring completely a polynomial when the number of degrees $n > 4$.

Is factorization unique? Yes, and No!

$$x^3 - x^2 = x^2(x-1) \quad \Rightarrow \quad \text{factors are } x^2 \text{ and } (x-1)$$

$$6x^3 - x^2 = 6x^2(x-1/6) = x^2(6x-1) \qquad \text{what are the factors?}$$

We can solve the problem by insisting on the factors have leading coefficient 1, and the leading coefficient of the original polynomial is written in front of the factors

$$6x^3 - x^2 = 6x^2(x-1/6)$$

Example

Show that the polynomial $P(x) = x^3 + 2x^2 - 3x - 10$ has $(x-2)$ as a factor. What are the other factors of the polynomial?

Solution We can use long division of $P(x)$ by the factor $(x-2)$

				x^3	x^2	x^1	x^0
(1)	D(x)	Q(x)	P(x)	x^3	$2x^2$	-3x	-10
(2)	x-2	x^2		x^3	$-2x^2$	0	0
(3)			(1)-(2)	0	$4x^2$	-3x	-10
(4)	x-2	4x		0	$4x^2$	-8x	0
(5)			(3)-(4) --> R(x)	0	0	5x	-10
(6)	x-2	5		0	0	5x	-10
(7)			(5)-(6) --> R(x)	0	0	0	0

The remainder after the long division is zero, $R(x) = 0$, therefore, $(x-2)$ is a factor and

$Q(x) = x^2 + 4x + 5$ is a irreducible quadratic function since

$$b^2 - 4ac = 16 - (4)(1)(5) = -4$$

$$P(x) = x^3 + 2x^2 - 3x - 10 = (x-2)(x^2 + 4x + 5)$$

This is in agreement with the Fundamental Theorem of Algebra.

MULTIPLE ROOTS

Example Find all real roots and their multiplicity of the polynomial

$$P(x) = (x-5)^3 (3x+4)^2 (x^2+2)^2 (x+\pi^2)^4$$

Solution

$x = 5$ multiplicity = 3

$x = -4/3$ multiplicity = 2

$x^2 + 2 = 0$ no real roots multiplicity = 2

$x = -\pi^2$ multiplicity = 4

degree of polynomial n = 3 + 2 + 2 + 2 + 4 = 13

Consider the polynomial $P(x)$ with the root α where $x = \alpha$ $(x-\alpha) = 0$ with multiplicity r, then, the polynomial dP/dx has multiplicity of $(r-1)$

$$P(x) = (x-\alpha)^r S(x)$$

$$dP/dx = r(x-\alpha)^{r-1} S + (x-\alpha)^r dS/dx = (x-\alpha)^{r-1} \left(rS + (x-\alpha) dS/dx \right)$$

$\Rightarrow$ multiplicity = $(r-1)$ for dP/dx

$(x-\alpha)$ is called a factor of $P(x)$ of **order** r.

COMPLEX NUMBERS

Another statement of the **Fundamental Theorem of Algebra**:

Every polynomial equation of degree n with complex coefficients has n roots in the complex numbers.

Note: a **real number** is also complex a number with its imaginary part equal to zero.

For all polynomials with **real coefficients**, complex roots with non-zero imaginary parts $y \neq 0$ always occur as **conjugate pairs**.

If $\alpha = x + i\,y$ then $\beta = x - i\,y$ is also a root.

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Polynomial: real coefficients degree n	Number of roots	Possible combinations (complex roots $y \neq 0$
1	1	1 real root
2	2	2 real roots or 1 conjugate pair
3	3	3 real roots or 1 real root and 1 conjugate pair
4	4	4 real roots or 2 real roots and 2 conjugate pairs or 2 conjugate pairs

When the degree is odd (1, 3, 5, …) there is at least one real root !

Example

Factorize the polynomials

$$P(z) = z^2 + 4z - 5 \qquad Q(z) = z^2 + 4z + 5 \qquad R(z) = z^2 + 4i\,z + 5$$

Solution

We need to find the roots of each polynomial. The degree of each polynomial is n =2, therefore in each case there are two roots.

$$P(z) = z^2 + 4z - 5 = 0 \quad \Rightarrow \quad z = 1 \quad z = -5$$

The roots are $\alpha = 1 \quad \beta = -5$ **both roots are real** and the factors are

$$(z-1) \quad (z+5) \qquad P(z) = (z-1)(z+5)$$

$$Q(z) = z^2 + 4z + 5 = 0$$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad a = 1 \quad b = 4 \quad c = 5$$

$$z = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$$

The two roots are **conjugate pairs** $\alpha = -2 + i \quad \beta = -2 - i$ and the factors are

$$(z+2-i) \quad (z+2+i) \qquad Q(z) = (z+2-i)(z+2+i)$$

$$R(z) = z^2 + 4i\,z + 5 = 0$$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad a = 1 \quad b = 4i \quad c = 5$$

$$z = \frac{-4i \pm \sqrt{-16 - 20}}{2} = -2i \pm 3i \quad z = i \quad z = -5i$$

The two roots do not form a conjugate pair because **all the coefficients are not real**

$\alpha = i \quad \beta = -5i$ and the factors are

$(z - i) \quad (z + 5i) \qquad R(z) = (z - i)(z + 5i)$

Roots and coefficients $\alpha = i \quad \beta = -5i \quad \alpha\beta = 5 = c/a \quad \alpha + \beta = -4 = -b/a$

Note: when the coefficients are not all real numbers then the roots do not have to form conjugate pairs.