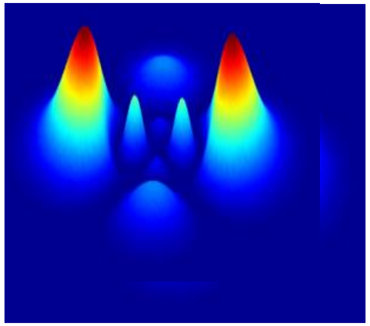

ADVANCED HIGH SCHOOL MATHEMATICS

POLYNOMIALS

ROOTS OF POLYNOMIALS

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Consider the polynomial equation of degree n

$$y = f(x) = P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n = \sum_{i=0}^{n} a_i x^i = 0$$

then real numbers x which satisfy this equation are called the **real roots** of the equation.

- For large values of $|x|$ then $y = f(x) \to a_n x^n$
- A polynomial where n is **odd** (**odd degree**) always has **at least one real root**
- At least one maximum or minimum value of the polynomial $f(x)$ occurs between any two distinct real roots.

$$y = f(x) = P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n = \sum_{i=0}^{n} a_i x^i = 0$$

This equation can be expressed as

$$P(x) = a_n (x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_{n-1})(x - \alpha_n) = \prod_{i=1}^{n} a_n (x - \alpha_i) = 0$$

The polynomial of degree n has n roots. Some of the roots maybe **real** and others may be **imagery**, also there may be **multiple** roots (eg $\alpha_2 = \alpha_3 = \alpha_6$) .

If α_i tis a real, then the term $(x - \alpha_i)$ is a **factor** of $P(x)$

Quadratic Equation

The graph of a quadratic function is a **parabola**. If there are real values of x for which $y = 0$, the parabola will intersect the X-axis at

real roots $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad b^2 - 4ac \geq 0$

Consider the quadratic equation

$$P(x) = a x^2 + b x + c = 0$$

Since the degree of the quadratic is $n = 2$, there are two roots which we designate as α and β. Hence we can express the polynomial as

$$P(x) = a\left(x - \alpha\right)\left(x - \beta\right) = 0$$

We can find the relationships between the coefficients a, b and c and the two roots α and β.

$$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{c}{a}\right) = 0$$

$$x^2 - \left(\alpha + \beta\right)x + \alpha\,\beta = 0$$

$$\alpha + \beta = -\frac{b}{a} \qquad \alpha\,\beta = \frac{c}{a}$$

Cubic Equation

Consider the cubic equation

$$P(x) = a\,x^3 + b\,x^2 + c\,x + d = 0$$

Since the degree of the cubic is n = 3, there are three roots which we designate as α , β and γ . Hence we can express the polynomial as

$$P(x) = a\left(x-\alpha\right)\left(x-\beta\right)\left(x-\gamma\right) = 0$$

We can find the relationships between the coefficients a, b, c and d and the three roots α , β and γ .

$$x^3 + \left(\frac{b}{a}\right)x^2 + \left(\frac{c}{a}\right)x + \left(\frac{d}{a}\right) = 0$$

$$x^3 - \left(\alpha+\beta+\gamma\right)x^2 + \left(\alpha\,\beta + \alpha\,\gamma + \beta\,\gamma\right)x - \alpha\,\beta\,\gamma = 0$$

$$\alpha+\beta+\gamma = -\frac{b}{a} \qquad \alpha\,\beta+\alpha\,\gamma+\beta\,\gamma = \frac{c}{a} \qquad \alpha\,\beta\,\gamma = -\frac{d}{a}$$

Quartic Equation

Consider the quartic equation

$$P(x) = a\,x^4 + b\,x^3 + c\,x^2 + dx + e = 0$$

Since the degree of the cubic is n = 4, there are four roots which we designate as α, β γ and δ. Hence we can express the polynomial as

$$P(x) = a\left(x-\alpha\right)\left(x-\beta\right)\left(x-\gamma\right)\left(x-\delta\right) = 0$$

We can find the relationships between the coefficients a, b, c and d and the three roots α ,β and γ.

$$x^4 \quad + \left(\frac{b}{a}\right)x^3 + \left(\frac{c}{a}\right)x^2 + \left(\frac{d}{a}\right)x + \left(\frac{e}{a}\right) = 0$$

$$x^4 - \left(\alpha+\beta+\gamma+\delta\right)x^3 + \left(\alpha\,\beta+\alpha\,\gamma+\alpha\,\delta+\beta\,\gamma+\beta\,\delta+\gamma\,\delta\right)x^2$$

$$-\left(\alpha\,\beta\,\gamma+\alpha\,\beta\,\delta+\alpha\,\gamma\,\delta+\beta\,\gamma\,\delta\right)x + \alpha\,\beta\,\gamma\delta = 0$$

$$\alpha+\beta+\gamma+\delta = -\frac{b}{a} \qquad \alpha\,\beta+\alpha\,\gamma+\alpha\,\delta+\beta\,\gamma+\beta\,\delta+\gamma\,\delta = \frac{c}{a}$$

$$\alpha\,\beta\,\gamma+\alpha\,\beta\,\delta+\alpha\,\gamma\,\delta+\beta\,\gamma\,\delta = -\frac{d}{a} \qquad \alpha\,\beta\,\gamma\,\delta = \frac{e}{a}$$

FACTORING

You have **factored** a polynomial if you write a polynomial as the product of two or more lower degree polynomials. For example:

$$2x^3 - 8x^2 - 3x + 12 = (x-4)(2x^2 - 3)$$

The polynomials $(x-4)$ and $(2x^2-3)$ are called **factors** of the polynomial $2x^3 - 8x^2 - 3x + 12$.

Note that the degrees of the factors 1 and 2, respectively, add up to the degree 3 of the polynomial we started with. Thus factoring breaks up a complicated polynomial into easier, lower degree pieces. We can also factor the polynomial $(2x^2-3)$

$$(2x^2-3) = 2(x^2 - 3/2) = 2\left(x - \sqrt{3/2}\right)\left(x + \sqrt{3/2}\right)$$

$$2x^3 - 8x^2 - 3x + 12 = 2(x-4)\left(x - \sqrt{3/2}\right)\left(x + \sqrt{3/2}\right)$$

We have now factored the polynomial completely into three linear polynomials (degree n = 1). Linear polynomials are the easiest polynomials.

Relationships between roots and coefficients of a polynomial

The roots of the polynomial $2x^3 - 8x^2 - 3x + 12$ are $\alpha = 4 \quad \beta = -\sqrt{3/2} \quad \gamma = +\sqrt{3/2}$

The general equation for a polynomial is $P(x) = a\,x^3 + b\,x^2 + c\,x + d = 0$ hence, in this example

$$a = 2 \quad b = -8 \quad c = -3 \quad d = 12$$

$$\alpha + \beta + \gamma = -\frac{b}{a} \quad \alpha\,\beta + \alpha\,\gamma + \beta\,\gamma = \frac{c}{a} \quad \alpha\,\beta\,\gamma = -\frac{d}{a}$$

$$\alpha + \beta + \gamma = 4 - \sqrt{3/2} + \sqrt{3/2} = 4 \quad -\frac{b}{a} = -\frac{(-8)}{2} = 4 \quad \color{blue}{QED}$$

$$\alpha\,\beta + \alpha\,\gamma + \beta\,\gamma = (4)\left(-\sqrt{3/2}\right) + (4)\left(\sqrt{3/2}\right) + \left(\sqrt{3/2}\right)\left(-\sqrt{3/2}\right) = -3/2 \quad \frac{c}{a} = \frac{-3}{2} \qquad \color{blue}{QED}$$

$$\alpha\,\beta\,\gamma = (4)\left(-\sqrt{3/2}\right)\left(\sqrt{3/2}\right) = -6 \quad -\frac{d}{a} = -\frac{12}{2} = -6 \quad \color{blue}{QED}$$

Finding a root $x = \alpha$ of a polynomial $P(x)$ is the same as having $\left(x - \alpha\right)$ as a linear factor of $P(x)$

$P(x) = \left(x - \alpha\right)Q(x)$ where n is the degree of $P(x)$ and (n-1) is the degree of $Q(x)$.