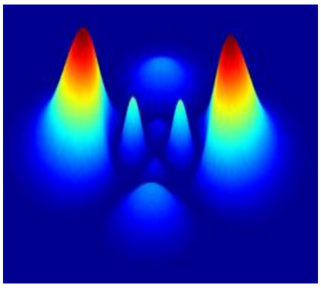

ADVANCED HIGH SCHOOL MATHEMATICS

MECHANICS

CONICAL PENDULUM

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An example of uniform circular motion is the **conical pendulum**. A conical pendulum has an object of mass *m* attached to the end of a string of length *L*. The object is rotated in a circle of radius *R* with a uniform speed *v* as shown in figure (1).

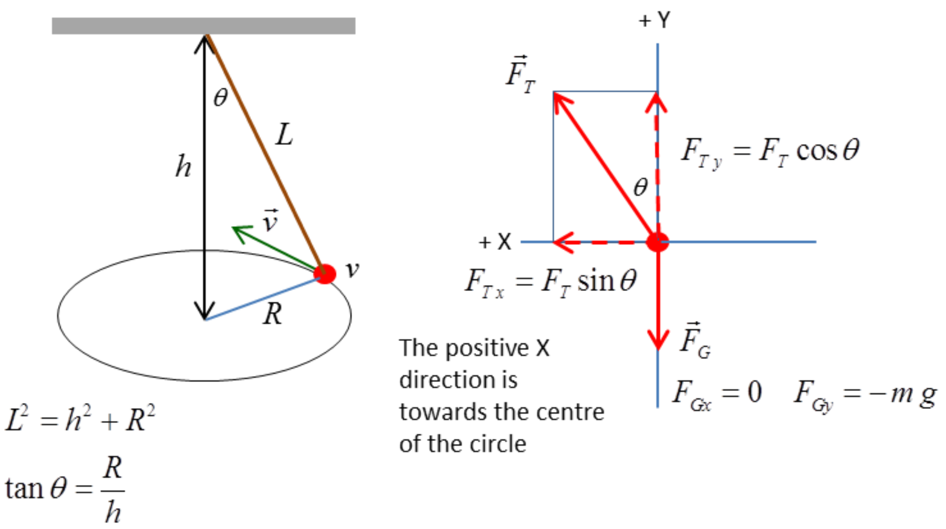


Fig. 1. A conical pendulum and free-body diagram showing the gravitation force $\vec{F}_G$ and the string tension $\vec{F}_T$ acting on the object of mass *m*.

The object has zero vertical acceleration and the horizontal acceleration is always directed towards the centre of the circle. Since the speed is uniform, the horizontal acceleration is the centripetal (radial or normal) acceleration. The resultant force in the horizontal direction which is directed towards the centre of the circle is the centripetal force. We can apply Newton's Second Law to the components of the forces acting in the X and Y directions.

Y direction

$$\sum F_y = F_T \cos\theta - F_G = 0 \qquad F_T = \frac{m\,g}{\cos\theta}$$

X direction

$$\sum F_x = F_T \sin\theta = \frac{m\,v^2}{R}$$

$$F_T = \frac{m\,v^2}{R\sin\theta}$$

Therefore, the angle θ the string makes with the vertical in executing uniform motion is determined from

$$\frac{m\,v^2}{R\sin\theta} = \frac{m\,g}{\cos\theta}$$

$$\tan\theta = \frac{v^2}{g\,R}$$

The uniform speed of the object is

$$v = \sqrt{g\,R\tan\theta}$$

The vertical distance h from the attachment of the string to the horizontal plane of motion is

$$L^2 = h^2 + R^2 \qquad \tan\theta = \frac{R}{h}$$

$$h = \sqrt{L^2 - R^2}$$

$$h = \frac{R}{\tan\theta}$$

$$h = \frac{g\,R^2}{v^2} \qquad v = R\sqrt{\frac{g}{h}}$$

The time for one complete revolution of the circle is the period T. The object travels around the circumference $2\pi R$ in a time interval equal to the period T at a constant speed v

$$T = \frac{2\pi R}{v} = \frac{2\pi L\sin\theta}{v} \qquad v^2 = \frac{R^2\,g}{h}$$

$$T = 2\pi\sqrt{\frac{h}{g}}$$

The frequency f of rotation and angular speed ω (angular velocity or angular frequency) are

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{h}}$$

$$\omega = 2\pi\,f = \sqrt{\frac{g}{h}}$$

$$h = \frac{g}{\omega^2}$$

The vertical depth h of the object is independent of the length of the string L and the mass of the object m. As the speed v of the object increases in its circular orbit, the object rises (smaller h).

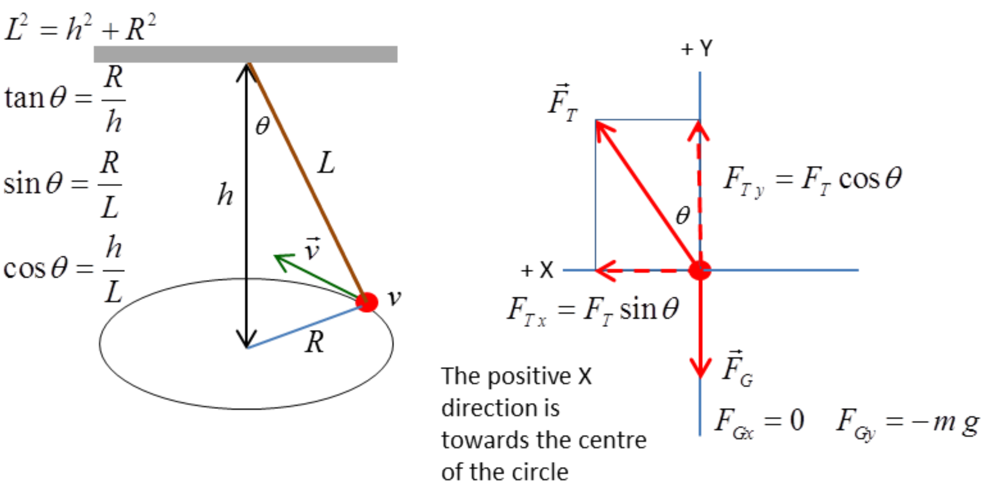
Example

A conical pendulum is formed by attaching a 300 g ball to a 1.00 m long string, then allowing the ball to move in a horizontal circle of radius 400 mm. Calculate

> Angle the string makes with the vertical θ [rad and degrees]
> String tension F_T [newtons N]
> Centripetal force F_C [N]
> Centripetal acceleration a_C [m.s^{-2}]
> Velocity of the ball v [m.s^{-1}]
> period of rotation T [s]
> frequency f [hertz Hz (s^{-1}) and revolutions per minute rpm]
> angular frequency (angular speed) ω [rad.s^{-1}]
> vertical depth of object h [m]

Solution

There are many ways in which you can calculate the unknown physical quantities. In this solution, the same quantities are often calculated in a number of ways. The first step in getting the answers is to construct an annotated diagram of the physical situation and converting all quantities to their S.I. unit.



$$R = 0.400 \text{ m} \quad L = 1.00 \text{ m} \quad m = 0.300 \text{ kg} \quad g = 9.8 \text{ m.s}^{-2}$$

Calculate θ

$$\sin\theta = \frac{R}{L} \qquad \theta = a\sin\left(\frac{R}{L}\right) = 0.412 \text{ rad} = 23.6^{\circ}$$

Calculate h

$$h = \sqrt{L^2 - R^2} = 0.917 \text{ m}$$

$$h = L\cos\theta = 0.917 \text{ m}$$

Calculate string tension F_T

$$F_T = \frac{m\,g}{\cos\theta} = 3.21 \text{ N}$$

Calculate centripetal force F_C

$$F_C = F_{Tx} = F_T \sin\theta = 1.28 \text{ N}$$

Calculate centripetal acceleration a_C

$$a_C = \frac{F_C}{m} = 4.28 \text{ m.s}^{-2}$$

Calculate velocity v

$$v = \sqrt{a_C\,R} = 1.31 \text{ m.s}^{-1}$$

$$v = \sqrt{g\,R\tan\theta} = 1.31 \text{ m.s}^{-1}$$

$$v = R\sqrt{\frac{g}{h}} = 1.31 \text{ m.s}^{-1}$$

Calculate Period T

$$T = \frac{2\pi R}{v} = 1.92 \text{ s}$$

$$T = 2\pi \sqrt{\frac{h}{g}} = 1.92 \text{ s}$$

Frequency f

$$f = \frac{1}{T} = 0.520 \text{ Hz}$$

1 rev in 1.92 s	1 s = (1/60) min
60 rev in 1.92 min	1 s^{-1} = 60 min^{-1}
(60/1.92) rpm	1 Hz = 60 rpm
f = 31.2 rpm	f = 0.520 Hz = 31.2 rpm

Angular speed (angular frequency) ω

$$\omega = 2\pi f = 3.27 \text{ rad.s}^{-1}$$

$$\omega = \sqrt{\frac{g}{h}} = 3.27 \text{ rad.s}^{-1}$$