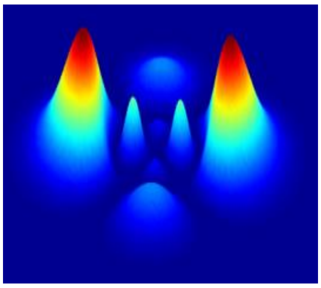

ADVANCED HIGH SCHOOL MATHEMATICS

MECHANICS

MOTION AROUND A BANKED CIRCULAR TRACK

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An example of centripetal forces and accelerations occurs when a car rounds a curve. As a passenger in the car, you feel a sensation of being thrust outward. But there is no force pulling you outward. What is happening is that you tend to move straight ahead, whereas, the car follows the bend in the road. The car must have an inward acting force on it to change its direction and travel around the curve. On a flat road, this force is due to the frictional force between road and tyres. Therefore, a force acts upon you due to friction with the seat or door to make you also go around the curve in the road.

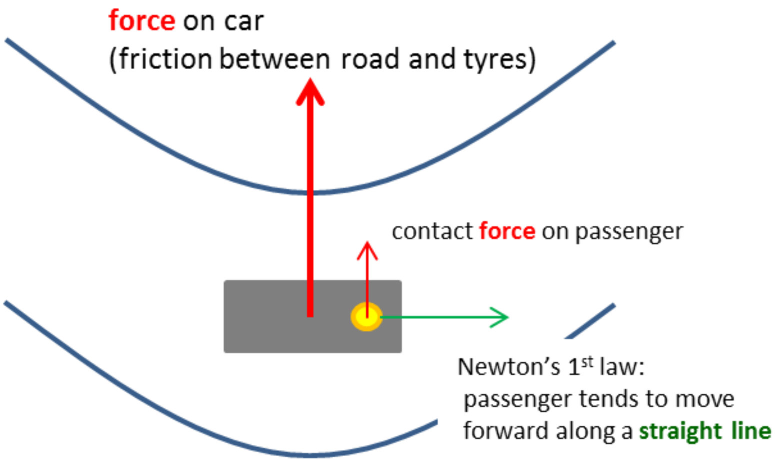


Fig. 1. The road exerts an inward force (friction: road and tyres) on the car to make it move in a circle and the car exerts an inward contact force on the passenger.

For the car going around a circular curve on a flat road we need to consider the forces acting on the car as shown in figure (2).

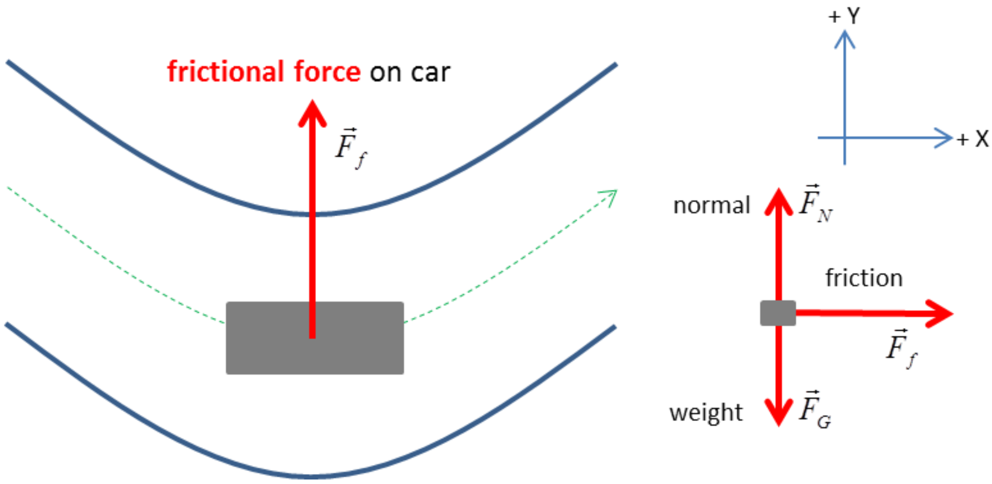


Fig. 2. Forces on a car rounding a circular bend on a flat road (top view and front view.

Applying Newton's Second Law to the car:

$$\vec{a} = \frac{1}{m}\sum_i \vec{F}_i$$

The car goes not leave the road

$$\sum F_y = F_N - F_G = 0 \qquad F_N = m\,g$$

The frictional force acts towards the centre and is the centripetal force causing the change in direction of the car.

$$\sum F_x = F_f = \frac{m\,v^2}{R}$$

If $\frac{m v^2}{R} > F_f$ then the car can't travel in a circle but will tend to follow a straighter path.

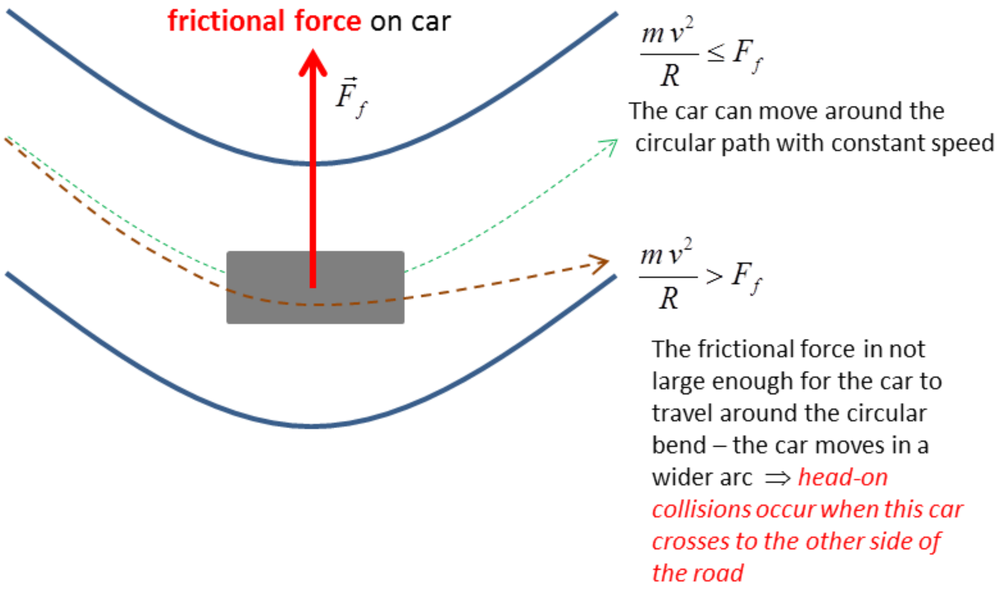


Fig. 3. A car travelling around a circular bend. There is a maximum speed at which the car can travel in a circular path. If this speed is exceeded it will move in a straighter (wider arc) path.

A simple model for the frictional force F_f is to assume that the frictional force is proportional to the normal force F_N and the constant of proportionality μ is called the **coefficient of friction**.

$$F_f = \mu F_N$$

For our flat road, $F_N = m g$ and $F_f = \mu m g$, hence the maximum speed v_{max} that a car can go around the circular bend is

$$v_{\max} = \sqrt{\mu g R}$$

Note: the maximum speed is independent of the mass *m* of the car.

However, having a **banked curve**, the maximum speed to travel in a circle can be dramatically increased even when the frictional force is zero because a component of the normal acts towards the centre and thus behaves as the centripetal force responsible for the change in direction.

Consider a car (mass m) travelling in a circular path (radius R) around a banked curve (angle θ) with a constant speed v as shown in figure (4) and assume that the frictional force acting on the car is zero, $F_f = 0$.

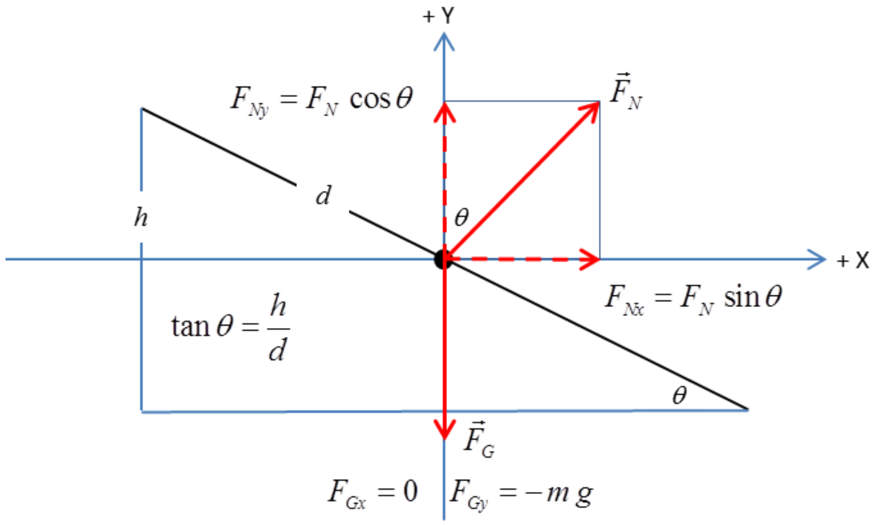


Fig. 4. Forces acting on a car rounding a banked curve.

The gravitational force and the normal force acting on the car can be resolved into their X and Y components. The centripetal acceleration is horizontal (parallel to the X axis). Apply Newton's Second Law to the X and Y components.

$$\sum F_x = F_N \sin\theta = \frac{m\,v^2}{R} \qquad \text{centripetal force}$$

$$\sum F_y = F_N \cos\theta - m\,g = 0 \qquad F_N = \frac{m\,g}{\cos\theta}$$

Eliminating F_N from these two equations gives

$$\tan\theta = \frac{v^2}{g\,R} \qquad v^2 = g\,R\tan\theta$$

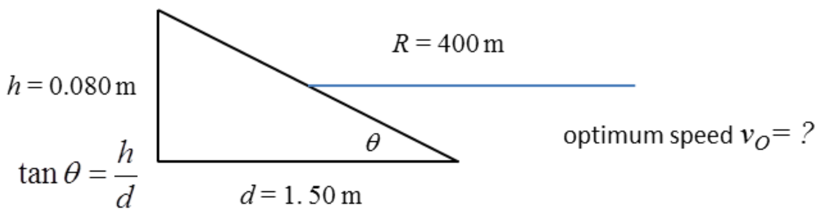
$$\tan\theta = \frac{h}{d} \qquad h = \frac{v^2\,d}{g\,R} \qquad v^2 = g\,R\frac{h}{d}$$

This means that the car can travel around the circular curve banked at the angle θ at speed v without any friction being required - there are zero lateral forces (forces parallel to the surface of the banked road) acting on the car. This is the **optimum speed** or **ideal speed** for the car to safely negotiate the curve.

The **banking angle** θ depends upon v and R but not the mass m. The larger the v the larger the banking angle needs to be and the smaller the banking angle the larger the radius of curvature of the curve.

Example

A circular curve of a railway track has a radius of 400 m. The distance between the rails is 1.50 m. The outside rail is 0.080 m above the inside rail. What is the optimum speed for a train to negotiate the curve so that the sideways force between the wheels and rail is minimized?



$$v_O = \sqrt{g\,R\tan\theta} = \sqrt{g\,R\frac{h}{d}}$$

$$v_0 = \sqrt{(9.8)(400)(0.08)/(1.5)} = 14\ \text{m.s}^{-1} = 52\ \text{km.h}^{-1}$$

When the banking angle, speed and radius satisfy $\tan\theta = v^2 / g\,R$, the car rounds the curve smoothly, with no tendency to slide outward or inward. If the speed of the car exceeds the optimum speed, then friction between the road and car will act down the surface of the road and this frictional force will have a horizontal component which increases the centripetal force to prevent the car sliding outward. If the car's speed is less than the optimum speed, then the frictional force acts up the bank of the road.

Suppose we consider a particular car going around a particular banked turn. The centripetal force needed to turn the car ($m\,v^2 / R$) depends on the speed of the car *v* (since the mass *m* of the car and the radius *R* of the turn are fixed) - more speed requires more centripetal force, less speed requires less centripetal force. The horizontal component of the normal force ($F_{Nx} = F_N \cos\theta$) is fixed (since the bank angle are fixed). So, it makes sense that we found one particular speed at which the centripetal force needed to turn the car equals the centripetal force supplied by the road. This is the optimum or ideal speed v_O at which the car the car will negotiate the turn - even if it is covered with perfectly-smooth ice. Any other speed *v*, will require a frictional force between the car's tyres and the road surface to keep the car from sliding up or down the embankment.

Speed of car greater than optimum speed $v > v_0$

If the speed of the car v is greater than the optimum speed v_0 for the turn then the horizontal component of the normal force will be less than the required centripetal force and the car will "want to" slide up the incline, away from the centre of the turn. The frictional force will oppose this motion and will act to pull the car down the incline, in the general direction of the centre of the turn. Therefore, the horizontal component of the frictional force adds to the horizontal component of the normal force to give the required centripetal force for the car to turn the curve without moving up or down the incline provide that

$$F_{Nx} + F_{fx} \leq m\,v^2 / R$$

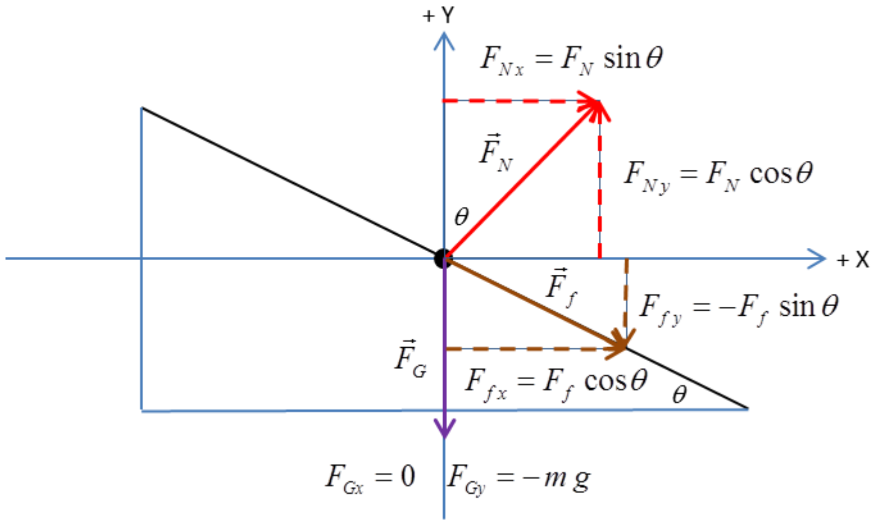


Fig. 5. Free-body diagram for the forces acting on car on banked circular track: weight, normal force and frictional force.

We can apply Newton's Second Law to the X and Y components of the forces:

$$\sum F_y = F_N \cos\theta - F_f \sin\theta - m\,g = 0$$
$$F_N \cos\theta - F_f \sin\theta = m\,g$$

$$\sum F_x = F_N \sin\theta + F_f \cos\theta = \frac{m v^2}{R}$$

We can use the approximation for the frictional force to relate it to the normal force

$$F_f = \mu F_N$$

We can now eliminate F_N and F_f from our equations

$$\sum F_y = F_N \cos\theta - F_f \sin\theta - m g = 0$$
$$F_N \cos\theta - F_f \sin\theta = m g$$
$$F_N \left(\cos\theta - \mu \sin\theta\right) = m g$$
$$F_N = \frac{m g}{\left(\cos\theta - \mu \sin\theta\right)}$$
$$F_N \sin\theta + F_f \cos\theta = \frac{m v^2}{R}$$
$$F_N \left(\sin\theta + \mu \cos\theta\right) = \frac{m v^2}{R}$$
$$F_N = \frac{m v^2}{R} \frac{1}{\left(\sin\theta + \mu \cos\theta\right)}$$
$$\frac{m g}{\left(\cos\theta - \mu \sin\theta\right)} = \frac{m v^2}{R} \frac{1}{\left(\sin\theta + \mu \cos\theta\right)}$$
$$v^2 = \left(g R\right)\left(\frac{\sin\theta + \mu \cos\theta}{\cos\theta - \mu \sin\theta}\right)$$
$$v^2 = \left(g R\right)\left(\frac{\tan\theta + \mu}{1 - \mu \tan\theta}\right)$$

If we take the case for $\mu = 0$ we get the same result again for the optimum speed $v^2 = \left(g R\right)\tan\theta$

Example

You want to negotiate a curve with a radius of 50 meters and a bank angle of 15°. If the coefficient of friction between the tyres and road surface is 0.50, what is the maximum speed that you can safely use? How does this compare with the optimum speed?

R = 65 m θ = 18° μ = 0.62 g = 9.8 m.s^{-2} v_{max} = ? m.s^{-1} v_O = ? m.s^{-1}

optimum speed $v_O^2 = (g\,R)\tan\theta$ max speed $v_{\max}^2 = (g\,R)\left(\dfrac{\tan\theta + \mu}{1 - \mu\tan\theta}\right)$

v_O = 14.4 m.s^{-1} v_{max} = 30.7 m.s^{-1}

$v_{max} > v_O$ as expected

Speed of car less than optimum speed $v < v_0$

If the speed of the car v is less than the optimum speed v_O for the curve then the horizontal component of the normal force will be greater than the required centripetal force and the car will "want to" slide down the incline toward the centre of the curve. If there is a frictional force present between the car's tyres and the road it will oppose this relative motion and pull the car up the incline.

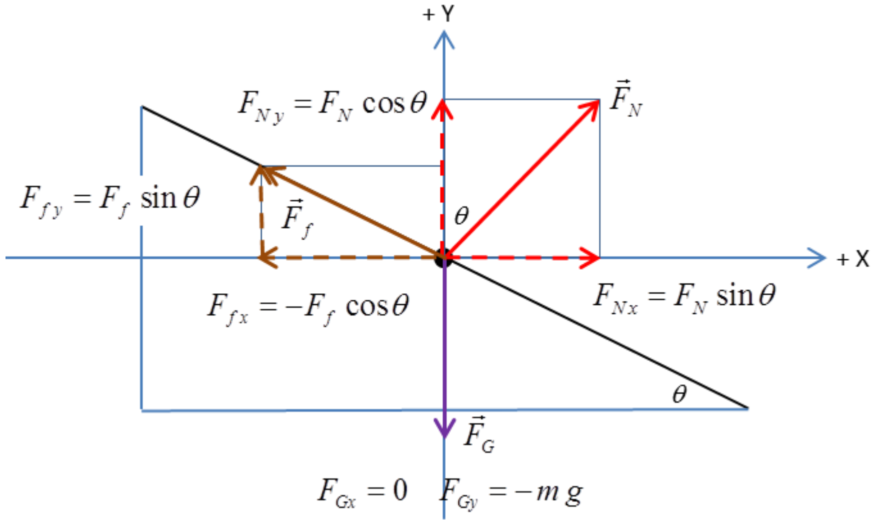


Fig. 6. Free-body diagram for the forces acting on car on banked circular track: weight, normal force and frictional force.

We can apply Newton's Second Law to the X and Y components of the forces:

$$\sum F_y = F_N \cos\theta + F_f \sin\theta - m\,g = 0$$
$$F_N \cos\theta + F_f \sin\theta = m\,g$$

$$\sum F_x = F_N \sin\theta - F_f \cos\theta = \frac{m\,v^2}{R}$$

We can use the approximation for the frictional force to relate it to the normal force

$$F_f = \mu\,F_N$$

We can now eliminate F_N and F_f from our equations

$$\sum F_y = F_N \cos\theta + F_f \sin\theta - m\,g = 0$$

$$F_N \cos\theta + F_f \sin\theta = m\,g$$

$$F_N \left(\cos\theta + \mu \sin\theta\right) = m\,g$$

$$F_N = \frac{m\,g}{\left(\cos\theta + \mu \sin\theta\right)}$$

$$F_N \sin\theta - F_f \cos\theta = \frac{m\,v^2}{R}$$

$$F_N \left(\sin\theta - \mu \cos\theta\right) = \frac{m\,v^2}{R}$$

$$F_N = \frac{m\,v^2}{R} \frac{1}{\left(\sin\theta - \mu \cos\theta\right)}$$

$$\frac{m\,g}{\left(\cos\theta + \mu \sin\theta\right)} = \frac{m\,v^2}{R} \frac{1}{\left(\sin\theta - \mu \cos\theta\right)}$$

$$v^2 = \left(g\,R\right)\left(\frac{\sin\theta - \mu \cos\theta}{\cos\theta + \mu \sin\theta}\right)$$

$$v^2 = \left(g\,R\right)\left(\frac{\tan\theta - \mu}{1 + \mu \tan\theta}\right)$$

If we take the case for $\mu = 0$ we get the same result again for the optimum speed $v^2 = \left(g\,R\right)\tan\theta$

Example

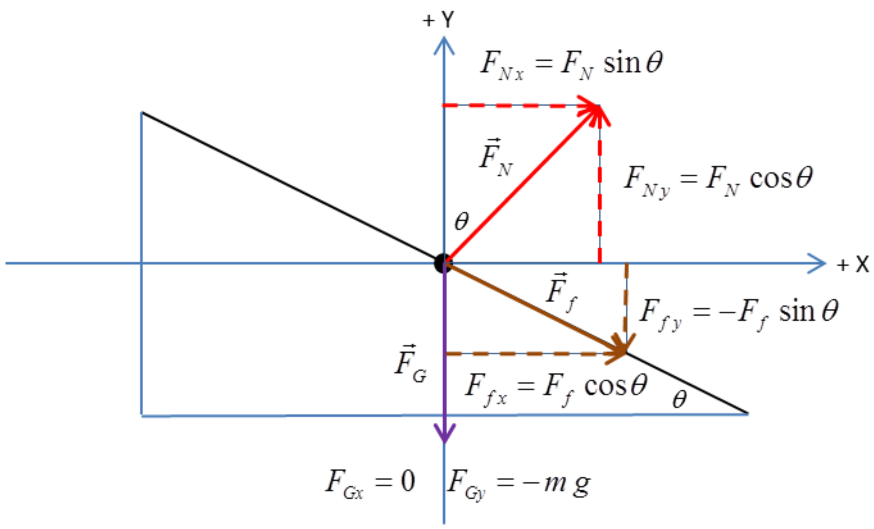
A car travels at speed v_1 around a circular curved track of radius R.

(a) Find the banked angle θ (inclination of the track to the horizontal), if there is to be no tendency for the car to slip sideways up or down the track.

(b) If the speed v_2 of a second car of mass m then show that the sideways frictional force exerted by the surface of the track on the wheels of the car is

$$F_f = m\,g\,\frac{\left(v_2^{\,2} - v_1^{\,2}\right)}{\sqrt{v_2^{\,4} + R^2 g^2}}$$

Solution



(a)

If the car goes around the curve without the need for friction, then, the car must travel at the optimum speed v_0, therefore $v_1 = v_0$

$$v_O^{\,2} = v_1^{\,2} = \left(g\,R\right)\tan\theta$$

The banking angle is determined from

$$\tan\theta = \frac{v_1^{\,2}}{g\,R}$$

(b)

Apply Newton's Second Law to the X and Y directions. From these two expressions containing terms involving the forces F_N and F_f, we can eliminate F_N to derive an equation for the frictional force F_f.

Y components

$$\sum F_y = F_N \cos\theta - F_f \sin\theta - m\,g = 0$$

$$F_N \cos\theta - F_f \sin\theta = m\,g$$

$$F_N = \frac{m\,g}{\cos\theta} + F_f \tan\theta \qquad \tan\theta = \frac{v_1^{\,2}}{g\,R}$$

$$F_N = \frac{m\,g^2\,R + F_f v_1^{\,2}\cos\theta}{g\,R\cos\theta}$$

X components

$$\sum F_x = F_N \sin\theta + F_f \cos\theta = \frac{m\,v_2^{\,2}}{R}$$

$$F_N \sin\theta = \frac{m\,v_2^{\,2}}{R} - F_f \cos\theta$$

$$F_N = \frac{m\,v_2^{\,2}}{R\sin\theta} - F_f\,\frac{1}{\tan\theta} \qquad \tan\theta = \frac{v_1^{\,2}}{g\,R}$$

$$F_N = \frac{m\,v_2^{\,2}}{R\sin\theta} - F_f\,\frac{g\,R}{v_1^{\,2}}$$

$$F_N = \frac{m\,v_1^{\,2}\,v_2^{\,2} - F_f\,g\,R^2\sin\theta}{R\,v_1^{\,2}\sin\theta}$$

Equating the two equations for F_N

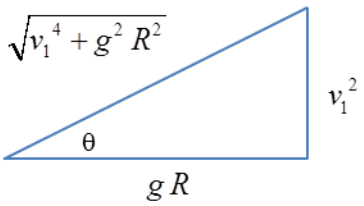
$$\frac{m\,g^2\,R + F_f v_1^{\,2}\cos\theta}{g\,R\cos\theta} = \frac{m\,v_1^{\,2}\,v_2^{\,2} - F_f\,g\,R^2\sin\theta}{R\,v_1^{\,2}\sin\theta}$$

$$\left(v_1^{\,2}\sin\theta\right)\left(m\,g^2\,R + F_f v_1^{\,2}\cos\theta\right) = \left(g\cos\theta\right)\left(m\,v_1^{\,2}\,v_2^{\,2} - F_f\,g\,R^2\sin\theta\right)$$

$$F_f\left(v_1^{\,4}\sin\theta\cos\theta + g^2\,R^2\sin\theta\cos\theta\right) = g\cos\theta\,m\,v_1^{\,2}\,v_2^{\,2} - m\,g^2\,R\,v_1^{\,2}\sin\theta$$

$$F_f\left(v_1^{\,4} + g^2\,R^2\right) = \left(m\,g\,v_1^{\,2}\right)\left(\frac{v_2^{\,2}}{\sin\theta} - \frac{g\,R}{\cos\theta}\right)$$

Now $\tan\theta = \dfrac{v_1^{\,2}}{g\,R}$ hence we can find expressions for $\sin\theta$ and $\cos\theta$



$$\tan\theta = \frac{v_1^{\,2}}{g\,R}$$

$$\sin\theta = \frac{v_1^{\,2}}{\sqrt{v_1^{\,4} + g^2\,R^2}}$$

$$\cos\theta = \frac{g\,R}{\sqrt{v_1^{\,4} + g^2\,R^2}}$$

$$F_f\left(v_1^{\,4} + g^2\,R^2\right) = \left(m\,g\,v_1^{\,2}\right)\left(\frac{v_2^{\,2}}{v_1^{\,2}} - \frac{g\,R}{g\,R}\right)\sqrt{v_1^{\,4} + g^2\,R^2}$$

$$F_f\left(\sqrt{v_1^{\,4} + g^2\,R^2}\right) = \left(m\,g\,v_1^{\,2}\right)\left(\frac{v_2^{\,2} - 1}{v_1^{\,2}}\right) = m\,g\left(v_2^{\,2} - v_1^{\,2}\right)$$

$$F_f = \frac{m\,g\left(v_2^{\,2} - v_1^{\,2}\right)}{\sqrt{v_1^{\,4} + g^2\,R^2}}$$

QED