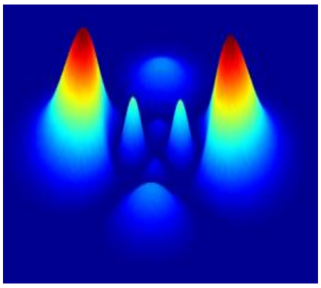

ADVANCED HIGH SCHOOL MATHEMATICS

MECHANICS

RESISTIVE MOTION IN THE HORIZONTAL DIRECTION

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Many problems in the mathematical analysis of particles moving under the influence of resistive forces, you start with the equation of motion. Then to find from the forces or acceleration you must then integrate the equation of motion to find velocities as functions of time and /or displacement and displacements as functions of time and /or velocity.

To be successful in this topic you need to be very familiar with the exponential and log functions and be competent in integrating functions.

BASIC KNOWLEDGE (you need to know this information very well)

$$\log_e(y) \equiv \ln(y)$$

$$\log_e(A_1\,A_2) = \log_e(A_1) + \log_e(A_2) \qquad \log_e(A_1 / A_2) = \log_e(A_1) - \log_e(A_2)$$

$$\log_e(A)^n = n\,\log_e(A)$$

$$y = A_1 e^{A_2 x} \qquad \log_e(y) \equiv \ln(y)$$

$$\log_e(y) = \log_e\left(A_1 e^{A_2 x}\right) = \log_e(A_1) + A_2\,x$$

$$\int \frac{dx}{a + b\,x} = \frac{1}{b}\log_e(a + b\,x) + C$$

$$\int \frac{f\,'(x)\,dx}{f(x)} = \log_e(f(x)) + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C = \frac{1}{a}\,a\tan\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log_e\left(\frac{a + x}{a - x}\right) + C$$

$$v(t) = \frac{dx(t)}{dt} \qquad x(t) = \int v(t)\,dt$$

$$a(t) = \frac{dv(t)}{dt} \qquad v(t) = \int a(t)\,dt$$

$$a(v) = \frac{dv(t)}{dt} = v\,\frac{dv(x)}{dx} \quad dx = \frac{v}{a(v)}\,dv \qquad x = \int \frac{v}{a(v)}\,dv$$

$$a(x) = \frac{dv(t)}{dt} = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right) \qquad d\left(\tfrac{1}{2}v^2\right) = \int a(x)\,dx$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} \qquad \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

$$\tan(A) = z \quad \mathrm{A} = \mathrm{atan}(\mathrm{z}) \equiv \tan^{-1}(z) \quad \tan^{-1}(\tan(A)) = A$$

MOTION UNDER THE INFLUENCE OF A RESISTIVE FORCE

In many mathematics textbook capital letters are used for forces, such as T for tension, G for gravity. In physics terms this not the best notation to use. The best use of symbols for force is the use F with a subscript to identify the type of force. For example

gravitational force (weight) FG

string tension F_T

resistive force F_R

Consider a particle of mass m acted upon by an applied force F_A and a resistive force F_R. The resistive force F_R is one that opposes the motion, i.e., the direction of the resistive force F_R is opposite to the velocity v of the particle. For one-dimensional motion, the direction of a vector is indicted as a positive or negative number with respect to a chosen frame of reference.

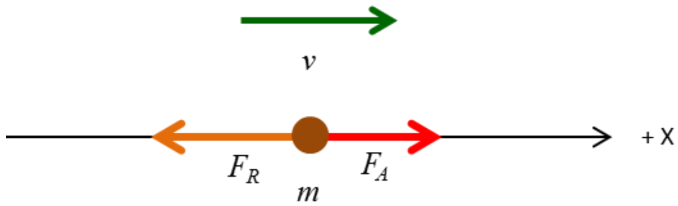


Fig. 1. Forces acting on a particles of mass m are an applied force F_A and a resistive force F_R opposing the motion of the object. The frame of reference is the + X axis pointing to the right.

The equation of motion of the particle can be derived from Newton's Second Law

(1) $$\vec{a} = \frac{1}{m}\sum_i \vec{F}_i$$ Newton's Second Law

(2) $$a = \frac{F_A - F_R}{m}$$

Usually the resistive force F_R is a function of velocity v. Two very important examples are

(3) $$F_R = -\beta v$$

(4) $$F_R = -\alpha v^2$$

Most HSC exam questions on resistive motion focus on "lots" of algebraic manipulations and not on physical interpretations of the motions. Your best approach to this Topic is to do many exercises to get experience in solving HSC style questions.

Example

An object on the surface of the water of a liquid is released at time $t = 0$ and immediately sinks. Let x be its displacement in metres in a downward direction from the surface of the liquid at time t seconds.

The equation of motion is given by

$$a = \frac{dv}{dt} = 10 - \frac{v^2}{40}$$

where v is the velocity of the object.

Show that $v = \dfrac{20\left(e^t - 1\right)}{e^t + 1}$ and $x = 20\log_e\left(\dfrac{400}{400 - v^2}\right)$

How far does the object sink in the first 4 s?

Solution

We can solve the problem in two different ways (1) by substitution / differentiation and (2) integration

<u>Approach 1 substitution</u>

(1) $$v = \frac{20\left(e^t - 1\right)}{e^t + 1}$$

(2) $$v^2 = 400\,\frac{\left(e^t - 1\right)^2}{\left(e^t + 1\right)^2}$$

(3) $$\frac{dv}{dt} = 10 - \frac{v^2}{40} = \frac{400 - v^2}{40}$$

Differentiate (1) w.r.t. t

$$v = \frac{20\left(e^t - 1\right)}{e^t + 1} = 20\left(e^t - 1\right)\left(e^t + 1\right)^{-1}$$

$$\frac{dv}{dt} = (20)\left(e^t\right)\left(e^t + 1\right)^{-1} + (20)\left(e^t - 1\right)\left(-e^t\right)\left(e^t + 1\right)^{-2}$$

$$\frac{dv}{dt} = (20)\left(e^t\right)\left(e^t + 1\right)^{-1}\left(1 - \left(e^t - 1\right)\left(e^t + 1\right)^{-1}\right)$$

$$\frac{dv}{dt} = (20)\left(e^t\right)\left(e^t + 1\right)^{-1}\left(\frac{\left(e^t + 1\right) - \left(e^t - 1\right)}{\left(e^t + 1\right)}\right)$$

$$\frac{dv}{dt} = (20)\left(e^t\right)\left(e^t + 1\right)^{-1}\left(\frac{\left(e^t + 1\right) - \left(e^t - 1\right)}{\left(e^t + 1\right)}\right)$$

(4) $$\frac{dv}{dt} = \frac{40\,e^t}{\left(e^t + 1\right)^2}$$

Substitute (2) into (3)

$$\frac{dv}{dt}=\frac{(10)}{\left(e^{t}+1\right)^{2}}\left(\left(e^{t}+1\right)^{2}-\left(e^{t}-1\right)^{2}\right)$$
$$\frac{dv}{dt}=\frac{(10)}{\left(e^{t}+1\right)^{2}}\left(e^{2t}+2e^{t}+1-e^{2t}+2e^{t}-1\right)$$

(5) $$\frac{dv}{dt}=\frac{(40)\left(e^{t}\right)}{\left(e^{t}+1\right)^{2}}$$ same result as equation (4)

Approach 2 integration

$$\frac{dv}{dt}=10-\frac{v^{2}}{40}=\frac{20^{2}-v^{2}}{40}$$
$$\frac{dt}{40}=\frac{dv}{20^{2}-v^{2}}$$
$$\int_{0}^{t}\frac{dt}{40}=\int_{0}^{v}\frac{dv}{20^{2}-v^{2}} \qquad \int\frac{dx}{a^{2}-x^{2}}=\frac{1}{2a}\log_{e}\left(\frac{a+x}{a-x}\right)$$
$$\frac{t}{40}=\left(\frac{1}{40}\right)\log_{e}\left(\frac{20+v}{20-v}\right)$$
$$e^{t}=\left(\frac{20+v}{20-v}\right)$$
$$e^{t}\left(20-v\right)=\left(20+v\right)$$
$$v\left(e^{t}+1\right)=20\left(e^{t}-1\right)$$
$$v=\frac{20\left(e^{t}-1\right)}{\left(e^{t}+1\right)}$$

QED

Find the equation for x

$$a = \frac{dv}{dt} = v\,\frac{dv}{dx} = 10 - \frac{v^2}{40} = \frac{400 - v^2}{40}$$

$$\frac{dx}{40} = \frac{v\,dv}{400 - v^2}$$

$$\int_0^x \frac{dx}{40} = \int_0^v \frac{v\,dv}{400 - v^2}$$

$$\frac{x}{40} = \left[\left(\frac{-1}{2}\right)\log_e\left(400 - v^2\right)\right]_0^v$$

$$x = -20\left[\log_e\left(400 - v^2\right) - \log_e\left(400\right)\right]$$

$$x = 20\,\log_e\left(\frac{400}{400 - v^2}\right)$$

QED

v and x at time t = 4 s

$$v = \frac{20\left(e^t - 1\right)}{e^t + 1} = \frac{20\left(e^4 - 1\right)}{e^4 + 1}$$

$$x = 20\log_e\left(\frac{400}{400 - v^2}\right)$$

$$400 - v^2 = 400 - 400\left(\frac{e^4 - 1}{e^4 + 1}\right)^2 = 400\left(1 - \frac{\left(e^4 - 1\right)^2}{\left(e^4 + 1\right)^2}\right)$$

$$400 - v^2 = 400\left(\frac{e^8 + 2e^4 + 1 - e^8 + 2e^4 - 1}{\left(e^4 + 1\right)^2}\right) = 400\left(\frac{4e^4}{\left(e^4 + 1\right)^2}\right)$$

$$\left(\frac{400}{400 - v^2}\right) = \left(\frac{\left(e^4 + 1\right)}{2e^2}\right)^2$$

$$x = 20\,\log_e\left(\frac{\left(e^4 + 1\right)}{2e^2}\right)^2$$

$$x = 40\log_e\left(\frac{e^4 + 1}{2e^2}\right)$$

QED

Example

A particle of unit mass and velocity v moves in a straight line against a resistive force F_R where

$$F_R = -\left(v + v^3\right)$$

Initially the particle is at the origin and travelling with a velocity v_0 in a positive direction.

(a) Show that the displacement x and velocity v are related by the equation

$$x = \operatorname{atan}\left(\frac{v_0 - v}{1 - v_0\, v}\right) \qquad \operatorname{atan} \equiv \tan^{-1}$$

(b) Show that the time t which has elapsed when the particle is travelling with velocity v is given by

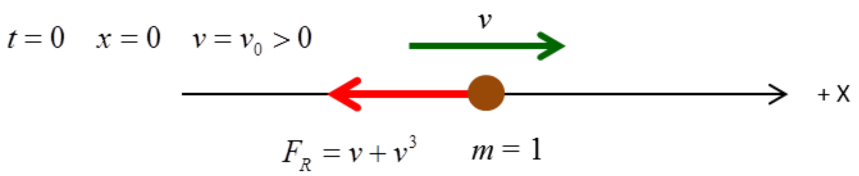
$$t = \tfrac{1}{2}\log_e\left[\frac{{v_0}^2\left(1+v^2\right)}{v^2\left(1+{v_0}^2\right)}\right]$$

(c) Find v^2 as a function of t.

(d) Finding the limiting values of v and x as $t \to \infty$.

Solution

(a)



Equation of motion $\quad a = \dfrac{dv}{dt} = v\dfrac{dv}{dx} = -\left(v + v^3\right)$

We can find x as a function of v by integrating the equation of motion. The integration limits are determined by the initial conditions and final values of x and v.

$$-dx = \frac{v\,dv}{v+v^3} = \frac{dv}{1+v^2}$$

$$-\int_0^x dx = \int_{v_0}^{v} \frac{dv}{1+v^2}$$

Using the standard integral $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C = \frac{1}{a}\,a\tan\left(\frac{x}{a}\right) + C$ $\quad a =$

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$$-x = \left[\operatorname{atan}(v)\right]_{v_0}^{v}$$

$$x = \operatorname{atan}(v_0) - \operatorname{atan}(v)$$

Let $A = \operatorname{atan}(v_0)$ and $B = \operatorname{atan}(v)$ and then take the tangent of x and using the facts that

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$$

$$\tan(A) = z \quad \text{A} = \operatorname{atan}(\text{z}) \equiv \tan^{-1}(z) \quad \tan^{-1}(\tan(A)) = A$$

$$\tan(x) = \frac{v_0 - v}{1+v_0\,v}$$

$$x = \operatorname{atan}\left(\frac{v_0 - v}{1+v_0\,v}\right)$$

QED

(b)

Equation of motion $a = \frac{dv}{dt} = -\left(v + v^3\right)$

We can find *v* as a function of *t* by integrating the equation of motion. The integration limits are determined by the initial conditions and final values of *t* and *v*.

$$a = \frac{dv}{dt} = -\left(v + v^3\right)$$

$$dt = \frac{-dv}{v + v^3} = \frac{-dv}{v\left(1 + v^2\right)} = -\left(\frac{1}{v} - \frac{v}{1 + v^2}\right) dv$$

$$\int_0^t dt = \int_{v_0}^{v} \left(\frac{v}{1 + v^2} - \frac{1}{v}\right) dv$$

$$t = \left[\tfrac{1}{2}\log_e\left(1 + v^2\right) - \log_e\left(v\right)\right]_{v_0}^{v}$$

$$t = \tfrac{1}{2}\left[\log_e\left(1 + v^2\right) - 2\log_e\left(v\right)\right]_{v_0}^{v}$$

$$t = \tfrac{1}{2}\left[\log_e\left(\frac{1 + v^2}{v^2}\right)\right]_{v_0}^{v}$$

$$t = \tfrac{1}{2}\left[\log_e\left(\frac{1 + v^2}{v^2}\right) - \log_e\left(\frac{1 + {v_0}^2}{{v_0}^2}\right)\right]$$

$$t = \tfrac{1}{2}\log_e\left(\frac{{v_0}^2\left(1 + v^2\right)}{v^2\left(1 + {v_0}^2\right)}\right)$$

(c) The can rearrange the expression for *t* as a function v^2 to find v^2

$$t = \tfrac{1}{2}\log_e\left(\frac{{v_0}^2\left(1 + v^2\right)}{v^2\left(1 + {v_0}^2\right)}\right)$$

$$e^{2t} = \frac{{v_0}^2\left(1 + v^2\right)}{v^2\left(1 + {v_0}^2\right)}$$

$$e^{2t}\ v^2\left(1 + {v_0}^2\right) = {v_0}^2\left(1 + v^2\right)$$

$$v^2\left(e^{2t}\left(1 + {v_0}^2\right) - {v_0}^2\right) = {v_0}^2$$

$$v^2 = \frac{{v_0}^2}{e^{2t}\left(1 + {v_0}^2\right) - {v_0}^2}$$

(d) $t \to \infty \quad e^{2t} \to \infty \quad v \to 0 \quad x \to \operatorname{atan}\left(v_0\right)$