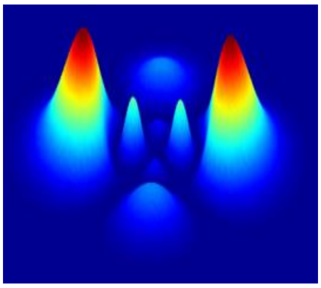


ADVANCED HIGH SCHOOL MATHEMATICS

MECHANICS

HARMONIC MOTION

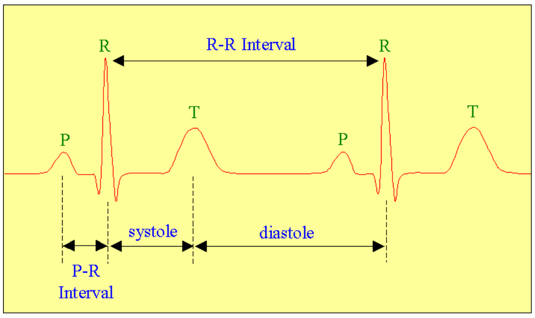
Ian Cooper email: matlabvisualphysics@gmail.com

Vibrations or oscillations are motions that repeated more or less regularly in time. The topic is very broad and diverse and covers phenomena such as mechanical vibrations (swinging pendulums, motion of a piston in a cylinder and vibrations of strings, rods, plates), sound, wave propagation, electromagnetic waves, AC currents and voltages.

Vibrations or oscillations are **periodic** if the values of physical quantities describing the motion are repeated in successive equal time intervals. The period T of vibration or oscillations is the minimum time interval in which all the physical quantities characterizing the motion are repeated. Thus, the period T is the time interval for one full vibration or cycle. The frequency f of periodic vibration is the number of vibration made per second.

(1) $$f = \frac{1}{T} \qquad T = \frac{1}{f}$$

VIEW some great animations on oscillation (UNSW Physclips)



What do these figures tell you about vibrations?

SIMPLE HARMONIC MOTION

The simplest type of periodic motion is called **simple harmonic motion** (**SHM**).

The displacement *x* of a particle executing SHM along the X axis is given by the sinusoidal function

(2) $$x = x_{max} \cos(\omega\, t + \phi)$$

x_{max} is the magnitude of the maximum displacement from the **equilibrium position** ($x = 0$). x_{max} is a positive number and is called the **displacement amplitude**

$\omega\, t + \phi$ called the **phase angle** [radians]

t is the time [s]

ω is the **angular frequency** [rad.s^{-1}]

ϕ is the **initial phase angle** (value of the phase angle at $t = 0$) [rad]. Its value determines the initial displacement of the particle $t = 0$,

$x = x_{max} \cos(\phi)$

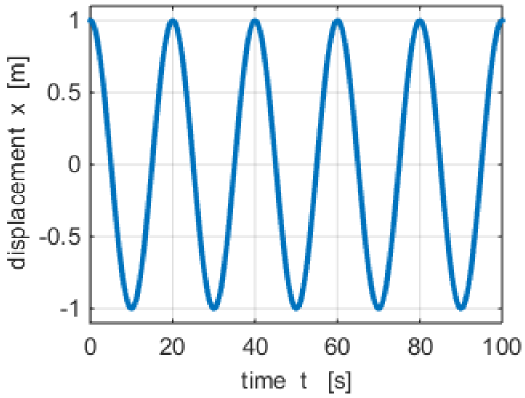
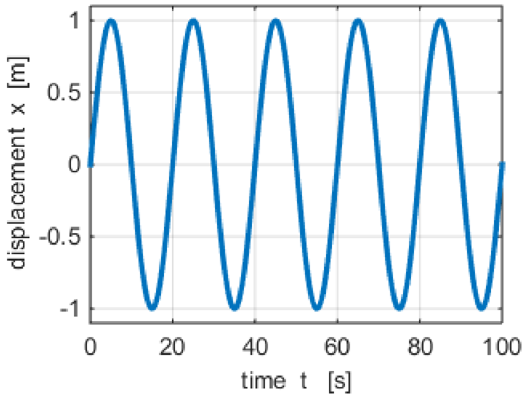
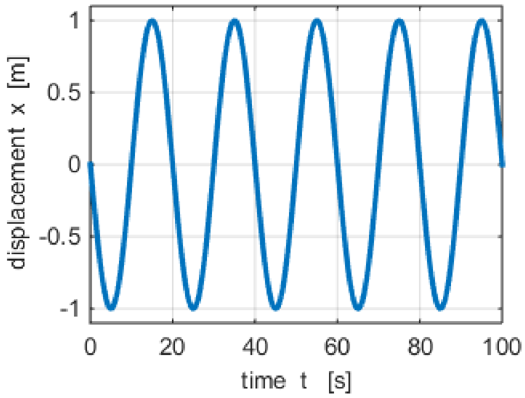
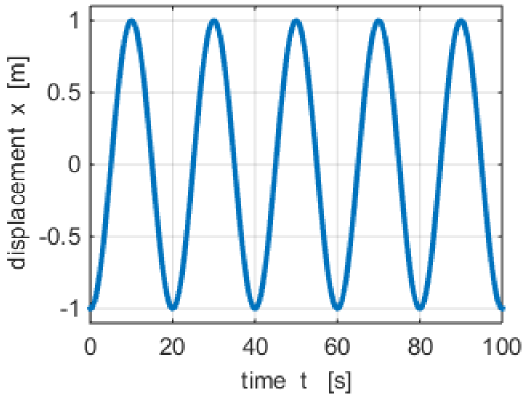
(3) $$\omega = 2\pi\, f = \frac{2\pi}{T}$$

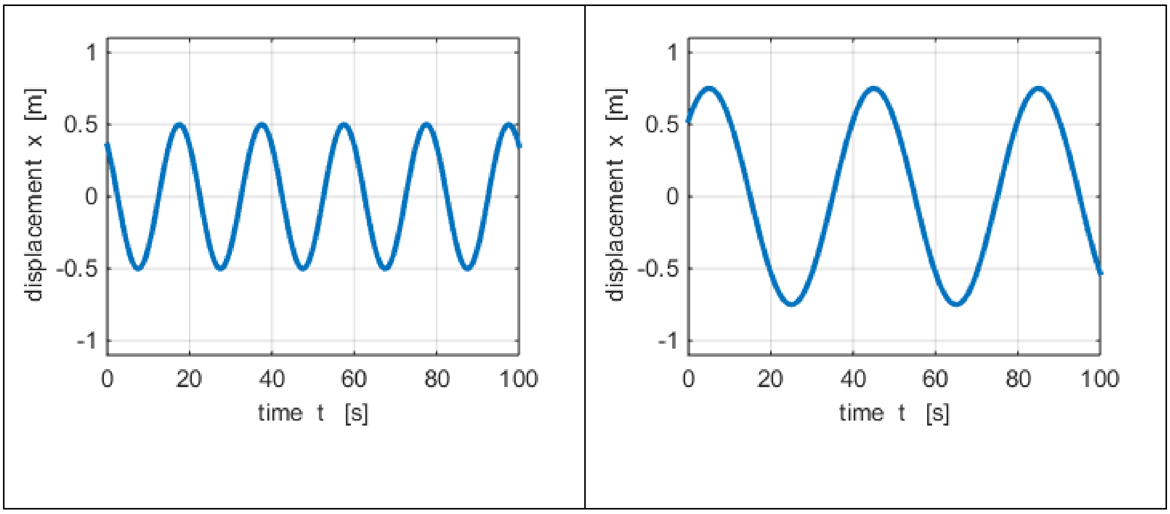
Equation (2) can also be written as:

A sine function $x = x_{max} \sin(\omega t + \phi')$

A sine and cosine function $x = A\cos(\omega t) + B\sin(\omega t)$

The values of the constants *xmax*, ϕ, ϕ', *A* and *B* can be determined from the initial conditions (*x* and *v* at time $t = 0$).

x_{max} = 1 m ϕ = 0 rad T = 20 s f = 0.05 Hz ω = 0.3142 rad.s^{-1}  	x_{max} = 1 m ϕ = -π /2 rad T = 20 s f = 0.05 Hz ω = 0.3142 rad.s^{-1} 
x_{max} = 1 m ϕ = π /2 rad T = 20 s f = 0.05 Hz ω = 0.3142 rad.s^{-1}  	x_{max} = 1 m ϕ = π rad T = 20 s f = 0.05 Hz ω = 0.3142 rad.s^{-1} 
x_{max} = 0.5 m ϕ = π / 4 rad T = 20 s f = 0.05 Hz ω = 0.3142 rad.s^{-1}	x_{max} = 0.75 m ϕ = - π / 4 rad T = 40 s f = 0.025 Hz ω = 0.1571 rad.s^{-1}



The **velocity** v is the time derivative of the displacement

$$
\begin{aligned}
&v = \frac{dx}{dt} = \dot{x} = \frac{d}{dt}\left[x_{\max}\cos(\omega\, t + \phi)\right] \\
(4)\qquad &v = -\left(x_{\max}\,\omega\right)\sin(\omega\, t + \phi) \\
&v = -v_{\max}\sin(\omega\, t + \phi) \qquad v_{\max} = x_{\max}\,\omega
\end{aligned}
$$

The displacement and velocity are π /2 rad out of phase with each other

$$x = 0 \quad \Rightarrow \quad v = \pm v_{\max} \qquad v = 0 \quad \Rightarrow \quad x = \pm x_{\max}$$

The velocity amplitude is $v_{\max} = x_{\max}\,\omega$ (always a positive number)

The **acceleration** a is the time derivative of the velocity

$$
\begin{aligned}
&a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \dot{v} = \ddot{x} = \frac{d}{dt}\left[-\left(x_{\max}\,\omega\right)\sin(\omega\, t + \phi)\right] \\
(5)\qquad &a = -\left(x_{\max}\,\omega^2\right)\cos(\omega\, t + \phi) \\
&a = -a_{\max}\cos(\omega t + \phi) \qquad a_{\max} = x_{\max}\,\omega^2 \\
&a = -\omega^2\, x
\end{aligned}
$$

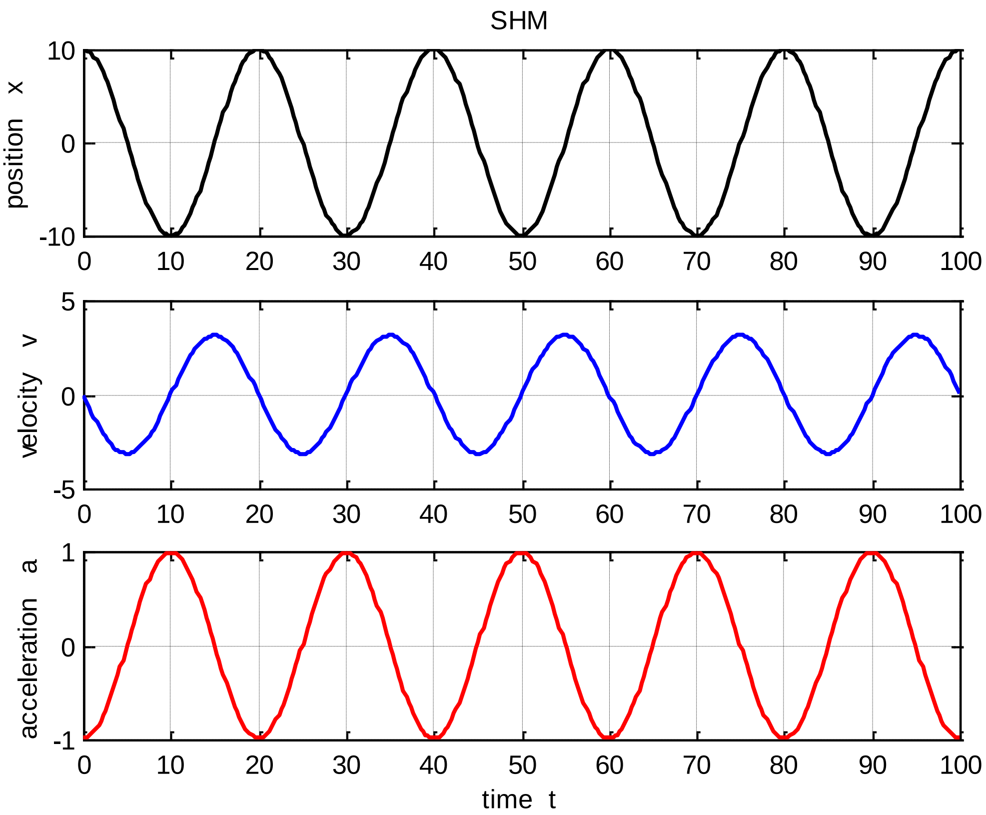
The acceleration amplitude is $a_{\max} = x_{\max}\,\omega^2$ (always a positive number)

The displacement and acceleration are π rad out of phase with each other

$$x = 0 \quad \Rightarrow \quad a = 0$$

$$x = +x_{\max} \quad \Rightarrow \quad a = -x_{\max}\,\omega^2 \qquad x = -x_{\max} \quad \Rightarrow \quad a = +x_{\max}\,\omega^2$$

The acceleration is always in the opposite direction to the displacement except at the equilibrium position ($x = 0 \quad a = 0$) and direction of the acceleration is directed towards the equilibrium position.



Since $a = -\omega^2 x$ the **equation of the motion** of the particle executing simple harmonic motion is

(6) $$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \ddot{x} + \omega^2 x = 0$$

Another approach to the mathematical analysis of SHM is to start with the equation of motion.

$$a = \frac{d^2x}{dt^2} = v\frac{dv}{dx}$$

The equation of motion then becomes

$$v\,dv = -\omega^2\,x\,dx$$

We can integrate this equation

$$\int v\,dv = \int\left(-\omega^2\,x\,dx\right)$$

$$\tfrac{1}{2}v^2 = \frac{-\omega^2}{2}x^2 + C'$$

$$v^2 = -\omega^2\,x^2 + C$$

where *C'* and *C* are constants which are determined from the initial conditions (t = 0).

Take the initial conditions to be $t = 0 \quad x = +x_{\max} \quad v = 0$

$$0 = -\omega^2\,x_{\max}{}^2 + C \qquad C = \omega^2\,x_{\max}{}^2$$

Therefore, the equation for the velocity as a function of displacement is

(7) $$v = \pm\omega\sqrt{x_{\max}{}^2 - x^2}$$

According to Newton's Second Law, an acceleration results from a non-zero resultant force acting on an object

(8) $$\vec{a} = \frac{1}{m}\sum_i \vec{F}_i$$

For SHM, the force acting on the particle is

(9) $$F = -m\,\omega^2\,x$$

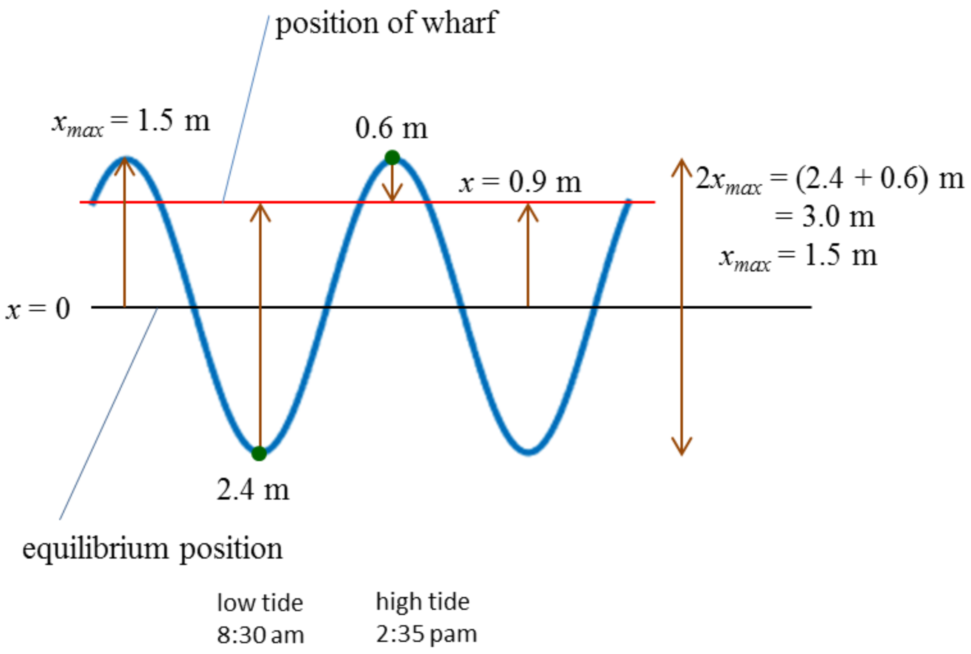
The resultant force *F* is always in the same direction as the acceleration *a*. The force responsible for SHM is called the **restoring force** and is always directed towards the equilibrium position (*x* = 0) and is proportional to the displacement.

Example (Syllabus)

The deck of a ship was 2.4 m below the level of the wharf at low tide and 0.6 m above the level at high tide. Low tide was at 8:30 am and high tide was at 2.35 pm. Find when the deck was level with the wharf, if the motion was simple harmonic.

Solution

The most important part of answering this question is constructing a good scientific diagram of the physical situation.



Take the initial conditions at the 8:30 am low tide

$t = 0$ s $v = 0$ m.s^{-1} $x = -x_{max} = -1.5$ m

The displacement as a function of time is

$$x = x_{\max} \cos(\omega\, t + \phi)$$

At time $t = 0$

$$x = x_{\max}\cos(\phi) = -x_{\max}$$
$$\cos(\phi) = -1$$
$$\phi = -\pi \text{ rad}$$

Hence

$$x = x_{\max}\cos(\omega\, t - \pi) = -x_{\max}\cos(\omega\, t)$$

The time interval from low tide to high tide is half-period $T/2$

8:30 am to 2:35 pm Δt = 6 hours 5 minutes = (6)(60)(60)+(5)(60) s = 21900 s

period T = 43800 s

angular velocity $\omega = \dfrac{2\pi}{T}$ = 1.4345x10^{-4} rad.s^{-1}

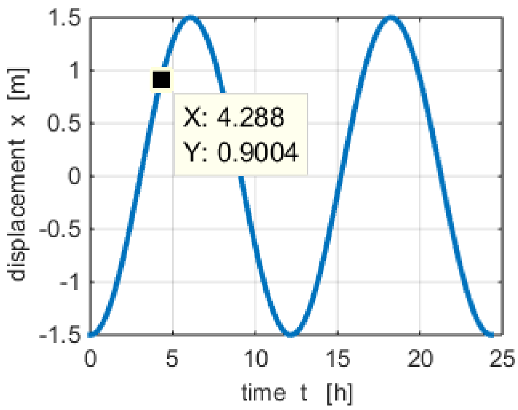
We want to find the time when the deck is level with the wharf

t = ? s x = 0.9 m position of wharf above equilibrium position (x = 0)

$$x = -x_{\max}\cos(\omega\, t)$$
$$\cos(\omega\, t) = \frac{-x}{x_{\max}}$$
$$\omega\, t = \operatorname{acos}\left(\frac{-x}{x_{\max}}\right)$$
$$t = \frac{\operatorname{acos}\left(\frac{-x}{x_{\max}}\right)}{\omega}$$

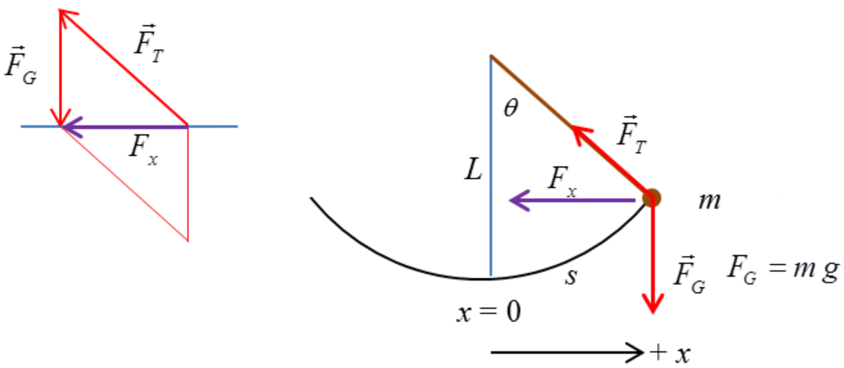
t = 1.5426x10^{4} s = 4.2877 h = 4 h 17 m

The deck will be level with the wharf at time 12:47 pm



SIMPLE PENDULUM

A simple pendulum is a particle of mass *m* suspended from a fixed point by a weightless, inextensible string of length *L*. It swings in a vertical plane. The forces acting on the particle are the gravitational force $\vec{F}_G$ and the string tension $\vec{F}_T$. For **small angle deviations** from the vertical, the motion of the particle is approximately SHM.



Applying Newton's Second Law to the particle of mass *m*

(8) $$\vec{a} = \frac{1}{m}\sum_i \vec{F}_i$$

We assume that the amplitude of the oscillation is small such that the resultant force only acts in the X direction

$$\sum \vec{F} = \vec{F}_G + \vec{F}_T = F_x \qquad F_y = 0$$

The assumption that $F_y = 0$ is only valid for small angles ($\theta < \sim 15^{o}$) and the following predictions do not give good agreement with measurements for large amplitude oscillations.

Adding the components in the Y direction gives

$$F_T \cos\theta = m\,g \qquad F_T = \frac{m\,g}{\cos\theta}$$

Adding the components in the X direction gives

$$F_x = -F_T \sin\theta = -\frac{m\,g}{\cos\theta}\sin\theta = -m\,g\tan\theta \qquad \tan\theta = \frac{x}{L}$$

$$F_x = -\left(\frac{m\,g}{L}\right)x$$

Therefore the acceleration a_x in the X direction is

(10) $$a_x = -\left(\frac{g}{L}\right)x$$ valid only for small values of x

But the acceleration a_x is opposite in direction to the displacement x and proportional to the displacement x , therefore, the motion of the particle is SHM.

Equation (5) for the acceleration of a particle executing SHM is

(5) $$a = -\omega^2\,x$$

Comparing equations (10) and (5), the angular frequency ω must be

(6) $$\omega = \sqrt{\frac{g}{L}}$$

hence, the period T of vibration and frequency f are

(7) $$T = 2\pi\sqrt{\frac{L}{g}} \qquad f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$ valid only for small values of x

The period T, frequency f and angular velocity ω only depend upon the length L of the pendulum's string and the acceleration due to gravity g, they do not depend upon the mass m of the particle or the **amplitude of oscillation**.

Be careful not to think that $\omega = \frac{d\theta}{dt}$ as in rotational (circular motion). Here θ is the angle of the pendulum at any instant. We now use ω not as the rate at which the angle θ changes, but rather as constant related to the period

$\omega = \frac{2\pi}{T} = \sqrt{\frac{g}{L}}$.

Example

Consider a simple pendulum of length $L = \frac{g}{\pi^2}$. The initial conditions for the vibration of the pendulum are $t = 0 \quad x = 0 \quad v = \pi$.

(a)

Find the first value of x where $v = 0$ by solving the equation of motion for the vibration of the pendulum.

(b)

The motion of the pendulum may more accurately be represented by the equation of motion

$$a_x = -\left(\frac{g}{L}\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)$$

Use this equation to find a more accurate answer for x than in part (a).

Solution

(a) $L = \frac{g}{\pi^2}$

Initial conditions $t = 0 \quad x = 0 \quad v = \pi$

Final conditions $x = ? \quad v = 0$

Equation of motion $a_x = \frac{d^2x}{dt^2} = v\frac{dv}{dx} = -\left(\frac{g}{L}\right)x$

Rearranging the equation of motion

$$v\,dv = -\left(\frac{g}{L}\right)x\,dx$$

Integrating this equation and using the initial condition and final conditions gives

$$\int_{\pi}^{0} v\,dv = -\frac{g}{L}\int_{0}^{x} x\,dx$$

$$\left[\tfrac{1}{2}v^2\right]_{\pi}^{0} = -\frac{g}{L}\left[\tfrac{1}{2}x^2\right]_{0}^{x}$$

$$-\pi^2 = \frac{g}{L}x^2$$

$$x = \pi\sqrt{\frac{L}{g}}$$

Note: the initial conditions give the lower limits and the final values give the upper limits for the integrations

(b)

$$a_x = v\frac{dv}{dx} = -\left(\frac{g}{L}\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)$$

$$\int_{\pi}^{0} v\,dv = -\left(\frac{g}{L}\right)\int_0^x \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)dx$$

$$-\tfrac{1}{2}\pi^2 = -\left(\frac{g}{L}\right)\left(\tfrac{1}{2}x^2 - \frac{x^4}{24} + \frac{x^6}{720}\right)$$

$$x^2 - \frac{x^4}{12} + \frac{x^6}{360} - \left(\frac{\pi^2 L}{g}\right) = 0$$

We need to find the value of x. We can find x by using **Newton's Method**

Newton's Method is a method for finding successively better approximations to the roots (or zeroes) of a real-valued function $f(x)$.

$x = ?\quad f(x) = 0$

We begin with a first guess x_1 for a root of the function $f(x)$. Then a better estimate of the root is approximated by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad f'(x) = \frac{d}{dx}f(x)$$

The process is repeated as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad f'(x) = \frac{d}{dx}f(x)$$

until a sufficiently accurate value is reached.

Let $$f(x) = x^2 - \frac{x^4}{12} + \frac{x^6}{360} - \left(\frac{\pi^2 L}{g}\right)$$

We can replace *x* by a variable *z* where $x = k\,z \quad k = \pi\sqrt{\dfrac{L}{g}} \quad z_1 = 1$

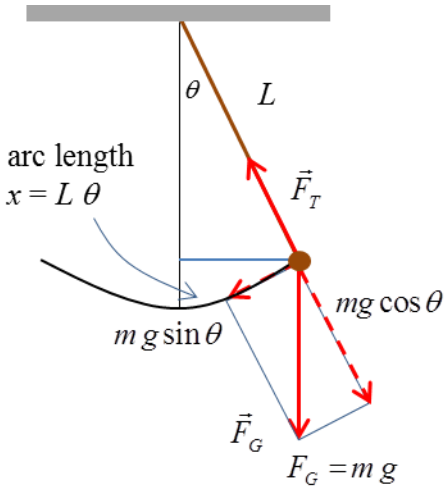
$$f(z) = k^2 x^2 - \frac{k^4\,x^4}{12} + \frac{k^6\,x^6}{360} - k^2$$
$$f\,'(z) = 2k^2 x - \frac{k^4\,x^3}{3} + \frac{k^6\,x^5}{60}$$

Take the first guess the solution given in part (a) $x_1 = \pi\sqrt{\dfrac{L}{g}} \quad z_1 = 1$

$$z_2 = z_1 - \frac{f(z_1)}{f\,'(z_1)} \quad z_1 = 1$$
$$f(1) = k^2 - \frac{k^4}{12} + \frac{k^6}{360} - k^2 = \frac{-30\,k^4 + k^6}{360}$$
$$f\,'(1) = 2k^2 - \frac{k^4}{3} + \frac{k^6}{60} = \frac{120\,k^2 - 20\,k^4 + k^6}{60}$$
$$z_2 = 1 - \left(\frac{-30\,k^4 + k^6}{360}\right)\left(\frac{60}{120\,k^2 - 20\,k^4 + k^6}\right)$$
$$z_2 = 1 - \frac{-30\,k^2 + k^4}{(60)\left(120 - 20\,k^2 + k^4\right)} \qquad x_2 = k\,z_2 \qquad k = \pi\sqrt{\frac{L}{g}}$$

hopefully answer is correct but it seems rather complicated – check carefully

Another look at the PENDULUM



The gravitational for force (weight) can be resolved into two components (1) tangent to the arc of the circle (2) perpendicular to the arc.

The particle of mass m moves forward and backward along the arc of the circle of radius L.

The displacement of the pendulum along the arc is given by

$$x = L\,\theta \qquad \theta \text{ must be in radians}$$

The restoring force F (force acting so that $\theta \to 0$) is the component of the gravitational force (weight $\vec{F}_G$ $F_G = m\,g$) tangent to the arc of the circle

$$F = -m\,g\,\sin\theta$$

where the minus sign means that the restoring force F is in the direction opposite to the angular displacement θ. F is proportional to $\sin\theta$ and not θ itself, hence the motion of the pendulum is **not** simple harmonic motion.

However, for small angular displacements ($\theta < 15^o$), the difference between the angle (in radians) and $\sin\theta$ is less than 1%.

$$\theta < 15^o \qquad \sin\theta \approx \theta$$

Therefore, for small angle oscillations of the pendulum

$$F = -m\,g\,\theta$$

the motion can be regarded as SHM.

Using the fact that the arc length *x* is given by $x = L\theta$, the restoring force and acceleration can be expressed as

$$F = -\frac{m\,g}{L}\,x \qquad F = m\,a$$

$$a = -\frac{g}{L}\,x$$

Again, we have the following relationships

Equation (5) for the acceleration of a particle executing SHM is

(5) $$a = -\omega^2\,x$$

Comparing equations (10) and (5), the angular frequency ω must be

(6) $$\omega = \sqrt{\frac{g}{L}}$$

hence, the period *T* of vibration and frequency *f* are

(7) $$T = 2\pi\sqrt{\frac{L}{g}} \qquad f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

For non-SHM the acceleration a of the pendulum is

$$a = -g\sin\theta$$

We can expand the function $\sin\theta$ in terms of the variable θ

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

Therefore, we can express the acceleration as

$$a = -g\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right)$$

It is now obvious that for small θ

$$a = -g\,\theta \qquad \text{SHM}$$