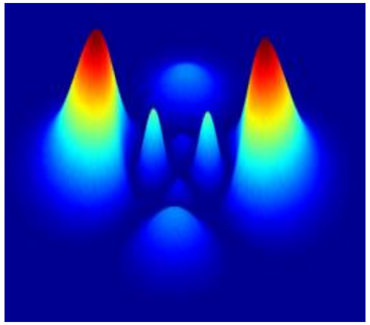
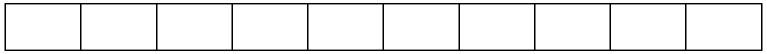


ADVANCED HIGH SCHOOL MATHEMATICS

PERMUTATIONS COMBINATIONS PROBABILITY

You should review Binomial Theorem before doing this topic.

How do we count the number of ways we can place the 10 numbers 0 1 2 3 4 5 6 7 8 9

into 10 boxes ? 

Any of the 10 numbers can be placed into the 1st box, 9 numbers into the 2nd box, 8 numbers into the 3rd box, and so on. So the total number of ways we can arrange all 10 numbers into the 10 boxes is

$$10! = (10)(9)(8)(7)(6)(5)(4)(3)(2)(1) = 3628800$$

The total number of ways of arranging n objects is called the number of **permutations** ${}^{n}P_{n}$

$${}^{n}P_{n} = n! = (n)(n-1)(n-2)\,\ldots\,(2)(1)$$

For example, the total number of permutations of the 26 letters of the English alphabet is ${}^{26}P_{26} = 26! = 4.032914611266057\;\ldots\;\times 10^{26}$ ***a very big number***

How do we count the number of ways we can place any five of the 10 numbers 0 1 2 3 4 5 6 7 8 9

into 5 boxes ?

Any of the 10 numbers can be placed into the 1st box, 9 numbers into the 2nd box, 8 numbers into the 3rd box, 7 numbers into the 4th box and 6 numbers into the 5th box. So the total number of ways we can arrange the numbers into the 5 boxes is $(10)(9)(8)(7)(6) = \dfrac{10!}{5!} = 30240$

Hence, the total number of arrangements or permutations of n distinct objects into k boxes is expressed as

$$^{n}P_{k} = (n)(n-1)(n-2)\ \ldots\ (n-k+1) = \frac{n!}{(n-k)!}$$

Example How many 6 letter number plates could be made if any of the 26 letters can be duplicated and if there is no repetition of any letter?

$\Rightarrow$ $n = 26$ $\quad k = 6$

No. of plates (repetitions allowed) = $(26)(26)(26)(26)(26)(26) = 26^6 = 308\,915\,776$

No. of plates (no repetitions) =

$$^{26}P_{6} = \frac{n!}{(n-k)!} = \frac{26!}{20!} = (26)(25)(24)(23)(22)(21) = 165\,765\,600$$

$\Leftarrow$

The number of ways of choosing k objects from n objects when the **order of the chosen objects matters is given by the number of permutations**

$$^{n}P_{k} = (n)(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

Example Twelve people are involved in a race. How many permutations (arrangements) are there for every possible winning combination of gold, silver and bronze medal winners?

$\Rightarrow$ $n = 12 \qquad k = 3$

No. of permutations (arrangements)

$$^{12}P_{3} = \frac{n!}{(n-k)!} = \frac{12!}{9!} = (12)(11)(10) = 1320$$

$\Leftarrow$

If the order in which the selected objects does not matter, then we can count the number of **combinations** that can occur.

The number of ways of choosing k objects from n objects when the order of the chosen items does **not** matter is

$${}^{n}C_{k} = \binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{P!}{k!} \quad \text{where } {}^{n}C_{k} \equiv \binom{n}{k} \text{ is a binomial coefficient.}$$

Example How many distinct arrangements are when 6 letter are selected from the 26 letters of the alphabet. How many combinations are there if the ordering of the letters is not important?

$\Rightarrow \quad n = 26 \qquad k = 6$

No. distinct arrangements =

$${}^{26}P_{6} = \frac{n!}{(n-k)!} = \frac{26!}{20!} = (26)(25)(24)(23)(22)(21) = 165\,765\,600$$

No. of combinations = $${}^{26}C_{6} = \frac{n!}{k!(n-k)!} = \frac{26!}{(6!)(20!)} = \frac{(26)(25)(24)(23)(22)(21)}{(6)(5)(4)(3)(2)(1)} = 230\,230$$

DGWJKS and GDJWSK are distinct (different) arrangements but the same combination – the six letters can be arranged in 6! (k!) ways

$$6!\left(\frac{26!}{(6!)(20!)}\right) = \frac{26!}{(20!)} \quad \Rightarrow \quad k!\,{}^{n}C_{k} = {}^{n}P_{k} \qquad \Leftarrow$$

Relationship between permutations and combinations for selecting k objects from n objects:

$$ {}^{n}C_k \equiv \binom{n}{k} = \frac{1}{k!}\,{}^{n}P_k \qquad {}^{n}P_k = k!\,{}^{n}C_k $$

Example I have a single pack of 52 cards. I draw a card, then draw a second card without putting the first card back in the pack. What is the probability that I draw two aces?

$\Rightarrow$ The number of ways of drawing 2 cards from 52 is

$$ {}^{n}C_k \equiv \binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad {}^{52}C_2 = \frac{52!}{2!\,(50)!} = \frac{(52)(51)}{(2)} = 1326 $$

The number of ways of getting two aces is the number of ways of drawing 2 aces from the 4 aces in the pack is

$$ {}^{4}C_2 = \frac{4!}{2!\,(2)!} = \frac{(4)(3)}{(2)} = 6 $$

♠♥ ♠♦ ♠♣ ♥♦ ♥♣ ♦♣

The probability that I draw two aces is therefore

$$ \text{Prob(2 aces)} = \frac{{}^{4}C_2}{{}^{52}C_2} = \frac{6}{1326} = \frac{1}{221} $$

Alternatively, $\text{Prob(2 aces)} = \left(\frac{4}{52}\right)\left(\frac{3}{51}\right) = \frac{1}{221}$ $\Leftarrow$

Example What is the probability of winning lotto where there are 40 balls and you have to match all six to win?

⇒ Method 1. $\text{Prob} = \left(\frac{6}{40}\right)\left(\frac{5}{39}\right)\left(\frac{4}{38}\right)\left(\frac{3}{37}\right)\left(\frac{2}{36}\right)\left(\frac{1}{35}\right) = \frac{1}{3\,838\,800} = 2.6053\times 10^{-7}$

Method 2. The number of ways in choosing 6 numbers from 40 where the order is not important is

$$^{40}C_6 = \frac{40!}{(6!)(34!)} = 3838380\,.$$

Therefore, the probability of winning lotto with one set of 6 numbers is

$$\text{Prob} = \frac{1}{^{40}C_6} = \frac{1}{3838380}$$

⇐

Example What is the probability choosing only one vowel when three letters are selected from the word **NUMBERS**? Assume each choice is equally likely.

$\Rightarrow$

Set of letters {N U M B E R S} $n = 7$

Subset of vowels {U E} $k = 2$ Subset of consonants {N M B R S} $m = 5$

Method 1. Think about the tree diagram for all the ways of selecting the 3 letters when there is only one vowel (v) and two consonants (c)

$$\text{Prob(v c c)} = (2/7)(5/6)(4/5)$$
$$\text{Prob(c v c)} = (5/7)(2/6)(4/5)$$
$$\text{Prob(c c v)} = (5/7)(4/6)(2/5)$$
$$\text{Prob(1 vowel)} = \text{Prob(v c c)} + \text{Prob(c v c)} + \text{Prob(c c v)}$$
$$\text{Prob(1 vowel \& 2 consonants)} = \frac{(2)(5)(4)+(5)(2)(4)+(5)(4)(2)}{(7)(6)(5)} = \frac{4}{7}$$

Method 2. The number of ways in choosing 3 letters from the 7 letters where the order is not important is ${}^{7}C_{3} = \frac{7!}{(3!)(4!)} = 35$.

The number of ways of choosing two vowels and five consonants is

$$ {}^{2}C_{1}\ {}^{5}C_{2} = (2)(5)(4)(3)(2)(1)/[(2)(3)(2)(1)] = 20$$

Therefore, the probability of choosing only vowel is $\text{Prob} = \frac{20}{35} = \frac{4}{7}$ $\Leftarrow$

Example How many numbers greater than 6000 can be found containing the digits 1, 3, 6, 7, 8 if no digit is repeated?

$\Rightarrow$

All 5 digit numbers are greater 6000 N_1 = (5)(4)(3)(2)(1) = 5! = 120

4 digit numbers must start with 6 or 7 or 8 N_2 = (3) (4)(3)(2)(1) = (3) 4! =72

Numbers > 6000 = $N_1 + N_2$ = 120 + 72 = 192 $\Leftarrow$

The number of **arrangements** or **permutations** on n items is $n!$ since the first position can be occupied by any of the *n* items, the second by any one of the remaining $(n-1)$ items and so on.

Permutations of n items ${}^nP_n = n!$

The number of **ordered** subsets containing k items out of n is by similar reasoning

Permutations of k items from n items

$${}^nP_k = n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!} \qquad k = n \quad 0! = 1 \quad {}^nP_n = n!$$

The number of **combinations** of n objects taken k at a time **without regard to the order** in which they appear is given by the number of permutations divided by $k!$ since each combination may be arranged in $k!$ ways

$${}^nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k} = {}^nC_{n-k}$$

For example, how many distinguishable arrangements are there of n items if n_1 of the items are identical, another n_2 are identical but different from the n_1 items, and so on. The total number of permutations is $n!$ but each distinguishable permutation appears $n_1!\,n_2!\ldots$ times, so that the number of distinguishable permutations is

$$\frac{n!}{n_1!\,n_2!\ldots}$$

Example In now many ways can the letters **A A A B B C D E E E** be arranged in a straight line?

$\Rightarrow$

Number of letters: $n = 10 \quad n_1 = 3 \quad n_2 = 2 \quad n_3 = 3$

Number of permutations $N = \dfrac{n!}{n_1!\,n_2!\ldots} = \dfrac{10!}{(3!)(2!)(3!)} = 50400$

$\Leftarrow$

Example Permutations Combinations Probability

A box contains 8 batteries, 5 of which are good and the other 3 are defective. Two batteries are selected at random and inserted into toy A and then another two batteries are selected and inserted into toy B. The toys only function with two good batteries.

(a) Calculate all the possible permutations of selecting 4 batteries from the 8 batteries.

(b) Calculate all the possible combinations of selecting 4 batteries from the 8 batteries.

(c) Calculate the probability that both toys function.

(d) Calculate the probability that only one of the toys function.

(e) Calculate the probability that both toys do not function.

$\Rightarrow$

A convenient way to attempt such problems is to use a spreadsheet for the setting out and performing the calculations. To find all the permutations you can use a tree-diagram, however, tree-diagrams are difficult to draw. A better approach is to make a modified tree-diagram showing all the branches as shown in the figure created in the spreadsheet. In the spreadsheet permutations and combinations are written without any superscripts or subscripts

$${}^{n}P_{k} \to nPk \qquad {}^{n}C_{k} \to nCk \, .$$

The total number of permutations of the eight batteries taken 4 at a time in a definite order is given by

$$ {}^{n}P_{k} = \frac{n!}{(n-k)!} \qquad {}^{8}P_{4} = \frac{8!}{4!} = (8)(7)(6)(5) = 1680 $$

since there 8 batteries to select from in the 1st choice; 7 batteries in the 2nd selection; and so on.

Let p represent a successful outcome of selecting a good battery and q represent a non-successful outcome when a defective battery is chosen. On each selection, there are only two outcomes: success or failure and there are 4 selections. Therefore, the number of branches or events in the modified tree-diagram is 2^4 = 16.

Figure P shows part of the spreadsheet for the calculations of the number of permutations and probability for different events. Consider the line showing branch 5 in the modified tree-diagram. The outcome of the selection of batteries is p q p p (batteries: good / defective/ good / good). 1st selection: 5 good batteries, 2nd selection: 3 defective batteries, 3rd selection: 4 good batteries; 4th selection: 3 good batteries to choose from. Therefore the number of permutations for this event is (5)(3)(4)(3) = 180. The probability of this event is

$$ P(pqpp) = \frac{\text{number of arrangements for pqpp}}{\text{total number of arrangements}} = \frac{180}{1680} = 0.1071 = \frac{3}{28} $$

The arrangement pqpp implies that toys A does not work (one defective battery) and toy B functions OK (two good batteries). Hence, for this event, the probability of only one toy functioning OK is 0.1071 = 3/28.

PERMUTATIONS

total number of permutations 8P5 = (8)(7)(6)(5) = 1680

Modified tree-diagram p (good battery) q (defectibe battery)

											Both toys OK		One toys only OK		Both toys not OK	
													pp/pq pp/qp pp/qq		pp/pq pp/qp pp/qq	
	Outcomes				**Number of selections**				Permutations	Probability	pp/pp		qq/pp qp/pp pq/pp		qq/pp qp/pp pq/pp	
	1st	2nd	3rd	4th	1st	2nd	3rd	4th	Perm	Prob	Perm	Prob	Perm	Prob	Perm	Prob
1	p	p	p	p	5	4	3	2	120	0.07143	120	0.0714				
2	p	p	p	q	5	4	3	3	180	0.10714			180	0.1071		
3	p	p	q	p	5	4	3	3	180	0.10714			180	0.1071		
4	p	p	q	q	5	4	3	2	120	0.07143			120	0.0714		
5	p	q	p	p	5	3	4	3	180	0.10714			180	0.1071		
6	p	q	p	q	5	3	4	2	120	0.07143					120	0.0714
7	p	q	q	p	5	3	2	4	120	0.07143					120	0.0714
8	p	q	q	q	5	3	2	1	30	0.01786					30	0.0179
9	q	p	p	p	3	5	4	3	180	0.10714			180	0.1071		
10	q	p	p	q	3	5	4	2	120	0.07143			120	0.0714		
11	q	p	q	p	3	5	2	4	120	0.07143					120	0.0714
12	q	p	q	q	3	5	2	1	30	0.01786					30	0.0179
13	q	q	p	p	3	2	5	4	120	0.07143			120	0.0714		
14	q	q	p	q	3	2	5	1	30	0.01786					30	0.0179
15	q	q	q	p	3	2	1	5	30	0.01786					30	0.0179
16	q	q	q	q	3	2	1	0	0	0.00000					0	0.0000
							sums -->		**1680**	**1.0000**	**120**	**0.0714**	**1080**	**0.6429**	**480**	**0.2857**

Fig. P. Part of the spreadsheet for permutations and probability calculations.

The probability that both toys function P_{two} = 0.0714 = 2/28 (event #1 p p p p).
Probability that one toy functions is P_{one} = 0.6429 = 18/28 (events #2 #3 #4 #5 #9 #10 #13)
Probability that both toys do not work is P_{zero} = 0.2857 = 8/28 (events #6 # 7 #8 #11 #12 #14 #15 #16)

$$P_{two} + P_{one} + P_{zero} = 2/28 + 18/28 + 8/28 = 1$$

The total number of combinations is

$$ {}^{n}C_{k} = \frac{{}^{n}P_{k}}{k!} = \frac{n!}{k!\,(n-k)!} \qquad {}^{8}C_{4} = \frac{8!}{4!\,4!} = 70 $$

The combinations are shown in figure C.

COMBINATIONS								
	binomial coefficients							
						5Ck	3Ck	5Ck * 3Ck
p	**p**	**p**	**p**	5C4		5	1	5
p	**p**	**p**	**q**	5C3	3C1	10	3	30
p	**p**	**q**	**q**	5C2	3C2	10	3	30
p	**q**	**q**	**q**	5C1	3C3	5	1	5
q	**q**	**q**	**q**		0	1	0	0
						sum -->		**70**

Fig. C. Section of the spreadsheet used to calculate the number of combinations for different events.

Binomial distribution

We can model experiments where there are precisely two outcomes which we classify as either success or failure. If the probability of a given event is p (success) and the probability of the event not occurring (failure) is $q = (1 - p)$, then the probability of the of the event occurring k times out of n trails is

$$\text{Prob}(k \text{ success from } n \text{ trails}) = \binom{n}{k} p^k\, q^{n-k} = {}^nC_k\, p^k\, q^{n-k} \qquad q = 1 - p$$

Example

An urn contains 9 black balls and 21 white balls. A ball is selected at random from the urn and replaced 10 times. What is the probability that (A) zero back balls and (B) 5 black balls are selected in the 10 trials.

$\Rightarrow$

black balls $n_B = 9$ white balls $n_w = 21$ balls $N = 30$ number of trails $n = 10$

For each selection $\text{Prob(black)} = p = \dfrac{n_B}{n_B + n_w} = \dfrac{9}{30} = \dfrac{3}{10}$ $\text{Prob(white)} = q = 1 - p = \dfrac{21}{30} = \dfrac{7}{10}$

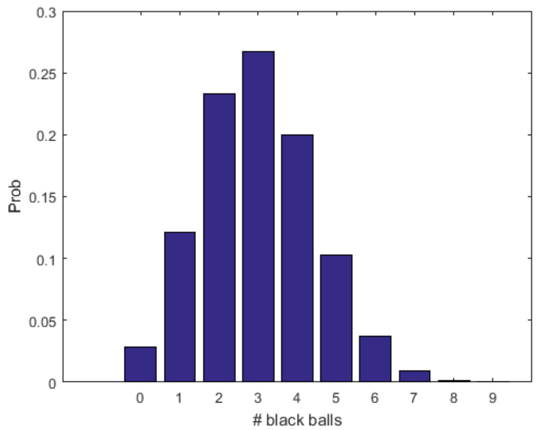
$$\text{Prob}(k \text{ success from } n \text{ trails}) = {}^nC_k\, p^k\, q^{n-k} \qquad {}^nC_k \equiv \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

(A) zero black balls chosen $k=0$

$$\text{Prob(0 success from 10 trails)} = {}^{10}C_0 \left(\frac{3}{10}\right)^0 \left(\frac{7}{10}\right)^{10} = 0.0282$$

(B) 5 black balls chosen $k=5$

$$\text{Prob(5 success from 10 trails)} = {}^{10}C_5 \left(\frac{3}{10}\right)^5 \left(\frac{7}{10}\right)^5 = 0.1029$$



Bar graph of the probability of selecting k black balls

⇐

If there n trails in the experiment, each with two possible outcomes then there are 2^n permutations of successes and failures. Consider the experiment of tossing a coin n times (or n coins simultaneously). The results of n successive coin flips can be represented by a sequence of n letters H (head) and T (tail).

In each coin toss, let H represent a success and T a failure. We can use the binomial distribution to predict the probability of k successes and $(n-k)$ failures where p is the probability of success and q the probability of failure

Probability of k successes $\text{Prob}(k \text{ success in } n \text{ trails}) = \binom{n}{k} p^k q^{n-k} = {}^nC_k \, p^k q^{n-k}$

For example, if $n = 5$ then there are $2^5 = 32$ distinct possible arrangements for the sequences of H and T. Assuming a fair coin, then the probability for each distinct sequence occurring is 1/32. One of the possible sequences is { H T H T T } and the probability of this sequence is 1/32.

A spreadsheet is a useful tool for investigating the sequences and probabilities of coin toss types of experiments. The following is a section of a spreadsheet for the tossing of a fair coin 5 times.

Probabilities: p success (H) q failure (T)

p = 0.5 q = 0.5

events	tosses -->					counts		probability
	1	2	3	4	5	H	T	$p^H * q^T$
1	H	H	H	H	H	5	0	0.03125
2	H	H	H	H	T	4	1	0.03125
3	H	H	H	T	H	4	1	0.03125
4	H	H	H	T	T	3	2	0.03125
5	H	H	T	H	H	4	1	0.03125
6	H	H	T	H	T	3	2	0.03125
7	H	H	T	T	H	3	2	0.03125
8	H	H	T	T	T	2	3	0.03125
9	H	T	H	H	H	4	1	0.03125
10	H	T	H	H	T	3	2	0.03125
11	H	T	H	T	H	3	2	0.03125
12	H	T	H	T	T	2	3	0.03125
13	H	T	T	H	H	3	2	0.03125
14	H	T	T	H	T	2	3	0.03125
15	H	T	T	T	H	2	3	0.03125
16	H	T	T	T	T	1	4	0.03125
17	T	H	H	H	H	4	1	0.03125
18	T	H	H	H	T	3	2	0.03125
19	T	H	H	T	H	3	2	0.03125
20	T	H	H	T	T	2	3	0.03125
21	T	H	T	H	H	3	2	0.03125
22	T	H	T	H	T	2	3	0.03125
23	T	H	T	T	H	2	3	0.03125
24	T	H	T	T	T	1	4	0.03125
25	T	T	H	H	H	3	2	0.03125
26	T	T	H	H	T	2	3	0.03125
27	T	T	H	T	H	2	3	0.03125
28	T	T	H	T	T	1	4	0.03125
29	T	T	T	H	H	2	3	0.03125
30	T	T	T	H	T	1	4	0.03125
31	T	T	T	T	H	1	4	0.03125
32	T	T	T	T	T	0	5	0.03125
						sum --->		**1.00000**

probability	
5H 0T	0.03125
4H 1T	0.15625
3H 2T	0.31250
2H 3T	0.31250
1H 4T	0.15625
0H 5T	0.03125
sum -->	**1.00000**

counts (# events)	
5H 0T	1
4H 1T	5
3H 2T	10
2H 3T	10
1H 4T	5
0H 5T	1
sum -->	**32**

Spreadsheet showing the 32 possible sequences for 5 tosses of a fair coin.

The probability of each sequence is exactly the same $P(\text{each distinct sequence}) = 1/32 = 0.03125$.

The number of sequences that has two heads and three tails is 10 (check spreadsheet by counting all the sequences with two heads and three tails). This number can be calculated from the binomial coefficient for counting the number of ways of selecting 2 objects from 5

$$^{5}C_{2} = \frac{5!}{2!\,3!} = 10$$

The probability of the event of two heads and three tails in any order is

$$P(\text{2H 3T}) = (10)\,(1/32) = 0.3125$$

since the probability of each sequence is (1/32) and there are 10 sequences with two heads and three tails.

The probability for two success (two heads) can also be calculated using the binomial distribution

$$n = 5 \quad k = 2 \quad p = q = 1/2$$

$$\text{Prob(2H 3T)} = \binom{5}{2} p^2 q^3 = {}^{5}C_{2}\, p^2 q^3 = (10)\left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 = 0.3125$$

Example

A factory produces bolts and it is known that the percentage of the bolts that are defective is *f*. In a random sample of 5 bolts, what is the probability than (A) zero bolts are defective, (B) precisely 1 bolt is defective and (c) at most one bolt is defective?

$\Rightarrow$

Let the probability of getting a non-defective bolt be p and the selection of a defective bolt be q, therefore

$$q = f / 100 \qquad p = 1 - q = 1 - f / 100$$

The binomial distribution

$$\text{Prob}(k \text{ success in } n \text{ trails}) = \binom{n}{k} p^k q^{n-k} = {}^nC_k \, p^k q^{n-k}$$

can be used to calculate the probabilities where $n = 5$.

(A) $k = 5$ (zero defective bolts)

$$\text{Prob(A)} = \binom{5}{5} p^5 q^0 = (1)(1 - f / 100)^5 = 10^{-10}(100 - f)^5$$

(B) $k = 4$ (two defective bolts)

$$\text{Prob(B)} = \binom{5}{4} p^4 q^1 = (5(1 - f/100)^5 (f/100) = 5 \times 10^{-10} f (100 - f)^4$$

(C) At most one defective bolt

$$\text{Prob}(C) = 1 - \text{Prob(A)} - \text{Prob(B)}$$

$$\text{Prob}(C) = 1 - 10^{-10}(100 - f)^5 - 5 \times 10^{-10} f (100 - f)^4$$

$$\text{Prob}(C) = 1 - 10^{-10}(100 - f)^4 \left[(100 - f) - 5\right] = 1 - 10^{-10}(100 - f)^4 (105 - f)$$

$\Leftarrow$

Example

In cricket for a six ball over, the average scoring shot occurs every third ball. Estimate how many six ball overs occur in a thousand overs in which there are precisely two scoring shots.

$\Rightarrow$

We want to find the probability of 2 scoring shots in a 6 ball over and the probability of scoring a shot is $p = 1/3$ and not scoring a shot is $q = 2/3$.

The binomial distribution

$$\text{Prob}(k \text{ success in } n \text{ trails}) = \binom{n}{k} p^k \, q^{n-k} = {}^{n}C_k \, p^k \, q^{n-k}$$

can be used to calculate the probability for $n = 6 \quad k = 2 \quad p = 1/3 \quad q = 2/3$.

$$\text{Prob} = \binom{6}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4 = 0.3292$$

In 1000 overs, therefore the estimated number of overs with exactly two scoring shots is

overs = $(1000)(0.3292) = 329$ (nearest integer)

$\Leftarrow$