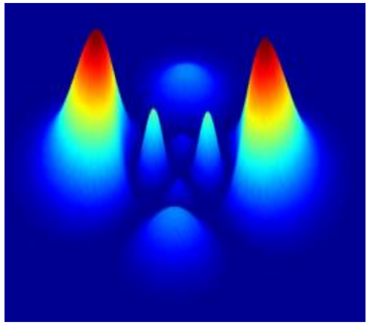

ADVANCED SCHOOL MATHEMATICS

BINOMIAL THEOREM

n – factorial $n!$

For positive integers n the product of the first n natural numbers is called n – factorial and is written

$$n! = (1)(2)(3)\ldots(n-1)(n)$$

$$0! = 1 \quad 1! = 1 \quad 2! = (1)(2) = 2 \quad 7! = (1)(2)(3)(4)(5)(6)(7) = 5040$$

A **binomial** is a polynomial of two terms such as $2x + 5y$ or $2 - (a+b)^3$.

The **Binomial Theorem** is a quick way of expanding a binomial expression that has been raised to some power n, for example, $(5x+6)^{12}$.

Consider the polynomial $(x+y)^n$ where n is an integer $n=1,\ 2,\ 3,\ \ldots$ This polynomial can be expanded using the **Binomial Theorem** in terms of the

binomial coefficients $\quad {}^{n}C_k \equiv \binom{n}{k} \qquad n=1,2,\ 3,\ldots \quad 0 \le k \le n$

$\binom{n}{k}$ is read as 'n over k'

$$ {}^{n}C_k = \binom{n}{k} = \frac{n!}{k!\left(n-k\right)!} \qquad 0 \le k \le n \qquad {}^{n}C_k = \binom{n}{k} = {}^{n}C_{n-k} = \binom{n}{n-k} = \frac{n!}{k!\left(n-k\right)!} $$

$$ {}^{n}C_0 = \binom{n}{0} = 1 \quad {}^{0}C_0 = \binom{0}{0} = 1 \quad {}^{n}C_n = \binom{n}{n} = 1 $$

$$ {}^{4}C_0 = \binom{4}{0} = 1 \quad {}^{4}C_1 = \binom{4}{1} = \frac{(4)(3)(2)(1)}{(1)(3)(2)(1)} = 4 = {}^{4}C_3 \quad {}^{4}C_2 = \binom{4}{2} = \frac{(4)(3)(2)(1)}{(1)(2)(2)(1)} = 6 $$

$$ {}^{4}C_3 = \binom{4}{3} = \frac{(4)(3)(2)(1)}{(1)(2)(3)(1)} = 4 = {}^{4}C_1 \quad {}^{4}C_4 = \binom{4}{4} = 1 $$

Pascal's triangle

An easy way to calculate the binomial coefficients is to use **Pascal's Triangle** where the binomial coefficients are arranged in a triangle. Each interior number is the sum of the nearest two numbers in the previous row.

1	${}^{0}C_0$
1 1	${}^{1}C_0 \; {}^{1}C_1$
1 **2** 1	${}^{2}C_0 \; {}^{2}C_1 \; {}^{2}C_2$
1 3 3 1	${}^{3}C_0 \; {}^{3}C_1 \; {}^{3}C_2 \; {}^{3}C_3$
1 **4 6 4** 1	${}^{4}C_0 \; {}^{4}C_1 \; {}^{4}C_2 \; {}^{4}C_3 \; {}^{4}C_4$
1 **5 10 10 5 1**	${}^{5}C_0 \; {}^{5}C_1 \; {}^{5}C_2 \; {}^{5}C_3 \; {}^{5}C_4 \; {}^{5}C_5$
1 6 **15** 20 15 **6** 1	${}^{6}C_0 \; {}^{6}C_1 \; {}^{6}C_2 \; {}^{6}C_3 \; {}^{6}C_4 \; {}^{6}C_5 \; {}^{6}C_6$

adding adjacent numbers gives the number in a row below e.g.

5 + **10** → **15**

$$\binom{6}{6}=1 \quad \binom{6}{5}=\frac{(6)(5)(4)(3)(2)(1)}{(5)(4)(3)(2)(1)(1)}=6 \quad \binom{6}{4}=\frac{(6)(5)(4)(3)(2)(1)}{(4)(3)(2)(1)(1)(2)}=15$$

$$\binom{6}{3}=\frac{(6)(5)(4)(3)(2)(1)}{(3)(2)(1)(1)(2)(3)}=20 \qquad \binom{6}{2}=\frac{(6)(5)(4)(3)(2)(1)}{(2)(1)(1)(2)(3)(4)}=15$$

$$\binom{6}{1}=\frac{(6)(5)(4)(3)(2)(1)}{(1)(1)(2)(3)(4)(5)}=6 \qquad \binom{6}{0}=1$$

The construction of a Pascal Triangle is based upon the relationship between binomial coefficients

$$^{n}C_{k}={}^{n-1}C_{k-1}+{}^{n-1}C_{k} \qquad 1\leq k\leq n-1$$

This is known as **Pascal's Triangle Identity**.

Example $LHS={}^{6}C_{2}=15 \quad RHS={}^{5}C_{1}+{}^{5}C_{2}=5+10=15 \quad \Rightarrow LHS=RHS$

Proof $\quad {}^{n}C_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k \qquad 1 \le k \le n-1$

$${}^{n}C_k = \frac{n!}{k!\,(n-k)!} \qquad 1 \le k \le n$$

$$k = 1 \quad {}^{n}C_1 = \frac{n!}{1!\,(n-1)!} = n \qquad {}^{n-1}C_0 + {}^{n-1}C_1 = 1 + \frac{(n-1)!}{1!\,(n-2)!} = 1 + n - 1 = n$$

QED

$k \ge 2$

$${}^{n-1}C_{k-1} + {}^{n-1}C_k = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{(k)!\,(n-k)!}$$

$$= \frac{(n-1)!}{(k-1)!\,(n-k)!}\left(\frac{1}{n-k} + \frac{1}{k}\right) = \frac{(n-1)!}{(k-1)!\,(n-k)!}\left(\frac{k+n-k}{(k)(n-k)}\right)$$

$$= \frac{n!}{k!\,(n-k)!} = {}^{n}C_k$$

QED

The coefficients of the variables x and y in the expansion of $(x+y)^n$ are called the binomial coefficients. The $(k+1)^{\text{th}}$ binomial coefficient of order n (n a positive integer) is

$${}^{n}C_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$(k+1)^{\text{th}}$ binomial coefficient

${}^{n}C_k \equiv \binom{n}{k}$ gives the number of combinations of n things k at a time.

Binomial Theorem

$$(x+y)^n = x^n + \binom{n}{1} x^{n-1}\, y + \binom{n}{2} x^{n-2}\, y^2 + \ldots + \binom{n}{n-1} x\, y^{n-1} + y^n$$

$$(x+y)^n = \sum_{k=0}^{k=n} \binom{n}{k} x^{n-k}\, y^k \equiv \sum_{k=0}^{k=n} {}^nC_k\, x^{n-k}\, y^k$$

Example Expand the binomial expression $(a+b)^6$

$\Rightarrow$ Construct a Pascal Triangle to find the binomial coefficients

$$(a+b)^6 = a^6 + 6a^5 b + 15a^4 b^2 + 20a^3 b^3 + 15a^2 b^4 + 6a\,b^5 + b^6$$

$\Leftarrow$

The sum of the binomial coefficients is equal to 2^n and is obtained by setting $x = y = 1$

$$n = 6 \quad (1+1)^6 = 2^6 = \sum_{k=0}^{k=n} \binom{n}{k} = 1+6+15+20+15+6+1 = 64$$

The binomial theorem is proved by **induction**. That is, it is shown to hold for $n = 1$, and further shown that if it holds for any given value of n then it also holds for the next higher value of n.

$$(x+y)^n = \sum_{k=0}^{k=n} \binom{n}{k} x^{n-k} y^k \qquad \text{Suppose} \quad (x+y)^n = \sum_{k=0}^{k=n} \binom{n}{k} x^{n-k} y^k \ \text{ is true.}$$

Now consider

$$\begin{aligned}(x+y)^{n+1} &= (x+y)(x+y)^n \\ &= (x+y)\sum_{k=0}^{n}\binom{n}{k} x^{n-k} y^k \\ &= \sum_{k=0}^{n}\binom{n}{k} x^{n+1-k} y^k + \sum_{j=0}^{n}\binom{n}{j} x^{n-j} y^{j+1} \\ &= \binom{n}{0}x^{n+1} + \sum_{k=1}^{n}\binom{n}{k} x^{n+1-k} y^k + \binom{n}{n}y^{n+1} + \sum_{j=0}^{n-1}\binom{n}{j} x^{n-j} y^{j+1}\end{aligned}$$

Let $j = k-1 \quad k = j+1$

$$\begin{aligned}(x+y)^{n+1} &= x^{n+1} + y^{n+1} + \sum_{k=1}^{n}\binom{n}{k} x^{n+1-k} y^k + \sum_{k=1}^{n}\binom{n}{k-1} x^{n+1-k} y^k \\ &= x^{n+1} + y^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k} + \binom{n}{k-1}\right] x^{n+1-k} y^k\end{aligned}$$

Using Pascal's Identity $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

$$(x+y)^{n+1} = x^{n+1} + y^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k} x^{n+1-k} y^k = \sum_{k=0}^{n+1}\binom{n+1}{k} x^{n+1-k} y^k \qquad \mathit{QED}$$

Example (a) Use the binomial theorem to expand $(2+3x)^5$.

(b) What is the 4th term in the expansion for increasing powers of x?

(c) Find the largest coefficient in the expansion.

(d) Expand $(2-3x)^5$

(e) Use the binomial theorem to **differentiate** the function $y=(2+3x)^5$

(f) Use the binomial theorem to **integrate** $y=(2+3x)^5$ from $x = 0$ to $x = 1$.

$\Rightarrow$ (a)

Always write down the binomial theorem to start answering the question

$$(x+y)^5 = \sum_{k=0}^{k=5} \binom{5}{k} x^{n-k} y^k$$

$$(2+3x)^5 = \sum_{k=0}^{k=5} \binom{5}{k} (2)^{5-k} (3)^k x^k$$

Use Pascal's triangle to give the binomial coefficients $\binom{5}{k} \rightarrow 1\ 5\ 10\ 10\ 5\ 1$

$$\left(2+3x\right)^5 = (1)(2^5)(1) + (5)(2^4)(3)x + (10)(2^3)(3^2)x^2 + (10)(2^2)(3^3)x^3 + (5)(2)(3^4)x^4 + (1)(3^5)x^5$$
$$= 32 + 240x + 720x^2 + 1080x^3 + 810x^4 + 243x^5$$

(b) increasing powers of *x:* $x^0 \quad x^1 \quad x^2 \quad x^3 \quad x^4 \quad x^5$ 4th term $\Rightarrow k = 3$

4th term $t_4 = \binom{5}{3}(2)^{5-3}(3)^3 x^3 = 1080\,x^3$ in agreement with part (a)

(c) Let the coefficients be represented by a_k. Since x and y are positive, we can check whether the coefficients are increasing or decreasing by considering the ratio $a_{k+1} / a_k \quad 0 \le k < n$

If $a_{k+1}/a_k > 1 \qquad a_{k+2}/a_{k+1} < 1 \qquad a_k < a_{k+1} > a_{k+2} \quad \Rightarrow \quad a_{k+1}$ is the largest coefficient

$$a_{k+1}/a_k = \frac{\binom{5}{k+1}\left(2^{5-k-1}\right)\left(3^{k+1}\right)}{\binom{5}{k}\left(2^{5-k}\right)\left(3^k\right)} = \frac{(5-k)}{(k+1)}\left(\frac{3}{2}\right) > 1 \Rightarrow \quad k < 2.6 \quad k = 2 \quad k+1 = 3$$

Therefore $a_3 = 1080$ is the largest coefficient in agreement with part (a).

(d) $(x+y)^5 = \sum_{k=0}^{k=5} \binom{5}{k} x^{n-k} y^k \qquad \left(2-3x\right)^5 = \sum_{k=0}^{k=5} \binom{5}{k} \left(2\right)^{5-k} (-1)^k \left(3\right)^k x^k$

$$\left(2-3x\right)^5 = (1)(2^5)(1) - (5)(2^4)(3)x + (10)(2^3)(3^2)x^2 - (10)(2^2)(3^3)x^3 + (5)(2)(3^4)x^4 - (1)(3^5)x^5$$

$$= 32 - 240x + 720x^2 - 1080x^3 + 810x^4 - 243x^5$$

(e)

$$y = \left(2+3x\right)^5 \qquad dy/dx = (5)(3)(2+3x)^4$$

$$(x+y)^4 = \sum_{k=0}^{k=4} \binom{4}{k} x^{n-k} y^k \quad \text{Pascal's triangle} \qquad n = 4 \quad \to 1\ 4\ 6\ 4\ 1$$

$$dy/dx = (15)\left(2^4 + (4)(2)^3(3x) + (6)(2)^2(3x)^2 + (4)(2)(3x)^3 + (3x)^4\right)$$

$$dy/dx = 240 + 1440x + 3240x^2 + 3240x^3 + 1215x^4$$

$$y = \left(2+3x\right)^5 = 32 + 240x + 720x^2 + 1080x^3 + 810x^4 + 243x^5$$

$$dy/dx = 240 + 1440x + 3240x^2 + 3240x^3 + 1215x^4 \qquad \textit{QED}$$

(f)

$$y=(2+3x)^5 \quad I=\int_0^1 (2+3x)^5\,dx$$

$$I=\left(\frac{1}{18}\right)\left[(2+3x)^6\right]_0^1=\left(\frac{1}{18}\right)\left(5^6-2^6\right)=864.5$$

$$y=(2+3x)^5=32+240x+720x^2+1080x^3+810x^4+243x^5$$

$$I=\int_0^1\left(32+240x+720x^2+1080x^3+810x^4+243x^5\right)dx$$

$$I=\left[32x+120x^2+240x^3+270x^4+162x^5+40.5x^6\right]_0^1$$

$$I=32+120+240+270+162+40.5=864.5 \quad QED$$

$\Leftarrow$

Example Find the coefficients of x^4 and x^3 in the expansion of $\left(2x-\frac{1}{x}\right)^6$

Check your answer by expanding the expression using the binomial theorem.

$\Rightarrow$

$$(x+y)^6=\sum_{k=0}^{k=6}\binom{6}{k}x^{6-k}y^k \qquad \text{Pascal's triangle} \qquad n=6 \quad \to 1\ \ 6\ \ 15\ \ 20\ \ 15\ \ 6\ \ 1$$

$$\left(2x-\frac{1}{x}\right)^6=\sum_{k=0}^{k=6}\binom{6}{k}(2)^{6-k}\left(x^{6-k}\right)(-1)^k x^{-k}=\sum_{k=0}^{k=6}\binom{6}{k}(-1)^k(2)^{6-k}\left(x^{6-2k}\right)$$

For the term in x^4 we require $6-2k=4 \quad \Rightarrow \quad k=1$, therefore, the coefficient $a_{k=1}$ is

$$a_{k=1} = \binom{6}{1}(2)^5(-1) = -(6)(32) = -192$$

For the term in x^3 we require $6-2k=3 \quad \Rightarrow \quad k=1.5$, k is not an integer, there is no term in x^3

$$\left(2x-\frac{1}{x}\right)^6 = 64x^6 - 192x^4 + 240x^2 - 160 + 60/x^2 - 12/x^4 + 1/x^6$$

⇐

Example Find the term in the expansion of $(3+5x)^{20}$ with the greatest coefficient.

$\Rightarrow$

Always write down the binomial theorem to start answering the question

$$(x+y)^{20} = \sum_{k=0}^{k=20} \binom{20}{k} x^{n-k} y^{k} \qquad (3+5x)^{20} = \sum_{k=0}^{k=20} \binom{20}{k} (3)^{20-k} (5)^{k} x^{k}$$

In the expansion of $(x+y)^n$ where x and y are both positive, then successive coefficients in the expansion get larger and then smaller. Therefore, the ratio R of the $(k+1)^{\text{th}}$ coefficient to the k^{th} coefficient will exceed one or be equal to one until the largest term is reached.

If $a_{k+1} / a_k > 1 \qquad a_{k+2} / a_{k+1} < 1 \qquad a_k < a_{k+1} > a_{k+2} \quad \Rightarrow \quad a_{k+1}$ is the largest coefficient

$$R = \frac{\dfrac{20!}{(k+1)!\,(20-k-1)!}\, 3^{20-k-1}\; 5^{k+1}}{\dfrac{20!}{k!\,(20-k)!}\, 3^{20-k}\; 5^{k}} = \left(\frac{5}{3}\right)\left(\frac{1}{k+1}\right)(20-k) \geq 1$$

$$100 - 5k \geq 3k + 3 \quad 8k \leq 97 \quad k \leq 12.125 \quad \Rightarrow \quad k = 12 \quad k+1 = 13$$

The largest term is term is $\binom{20}{13} 3^{7}\; 5^{13}\; x^{13}$

$\Leftarrow$

Example

(a) Expand $(x+1)^n$ substitute $x=1$

(b) Expand $(x-1)^n$ substitute $x=1$ hence show that

$$^nC_1 + ^nC_3 + ^nC_5 + \dots = ^nC_0 + ^nC_2 + ^nC_4 + \dots$$

(c) Show that $2^{n-1} = ^nC_1 + ^nC_3 + ^nC_5 + \dots = ^nC_0 + ^nC_2 + ^nC_4 + \dots$

(d) Prove $^{n+1}C_k = ^nC_k + ^nC_{k-1}$ which gives the Pascal's triangle identity

(e) Show that $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

$\Rightarrow$

Always write down the binomial theorem to start answering the question

$$(x+y)^n = \sum_{k=0}^{k=n} \binom{n}{k} x^{n-k} y^k$$

(a) $(x+1)^n = \sum_{k=0}^{k=n} \binom{n}{k} x^{n-k}$ $\qquad x=1 \Rightarrow 2^n = \sum_{k=0}^{k=n} \binom{n}{k} = ^nC_0 + ^nC_1 + ^nC_2 + \dots + ^nC_n$

(b) $(x-1)^n = \sum_{k=0}^{k=n} \binom{n}{k} (-1)^k x^{n-k}$

$$x = 1 \quad \Rightarrow \quad 0 = \sum_{k=0}^{k=n} \binom{n}{k} (-1)^k = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 \ldots + (-1)^n \, {}^nC_n$$

$${}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = {}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots$$

(c) Add the results from parts (a) and (b)

$$2^n = 2\left({}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots \right) \quad \Rightarrow$$

$$2^{n-1} = {}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots$$

(d) $(x+1)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k}$ coefficient for x^{n+1-k} is $\binom{n+1}{k}$

$$(x+1)^{n+1} = (x+1)(x+1)^n = (x+1)\sum_{k=0}^{n} \binom{n}{k} x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^{n+1-k} + \sum_{k=0}^{n} \binom{n}{k} x^{n-k}$$

coefficient for x^{n+1-k} is $\binom{n}{k} + \binom{n}{k-1}$

Therefore, ${}^{n+1}C_k = {}^nC_k + {}^nC_{k-1}$ $\qquad \binom{n}{k} \equiv {}^nC_k$

(e) $(x+1)^{2n} = \sum_{k=0}^{k=2n} \binom{2n}{k} x^{2n-k}$

We are interested in the term $\binom{2n}{n} x^n$ when $k = n$.

Also, $(x+1)^n = \sum_{k=0}^{k=n} \binom{n}{k} x^{n-k}$ $\quad (x+1)^n (x+1)^n = \sum_{k=0}^{k=n} \binom{n}{k} x^{n-k} \sum_{k=0}^{k=n} \binom{n}{k} x^{n-k}$

The term in x^n from the expansion expressed as a product is

$$\sum_{k=0}^{n} \left[\binom{n}{k} x^{n-k} \binom{n}{n-k} x^k \right]$$

x^n is formed by taking the product of terms in x^{n-k} from the first expression and x^k from the second expression in the product.

$$\binom{n}{k} = \binom{n}{n-k} \qquad \sum_{k=0}^{n} \left[\binom{n}{k} x^{n-k} \binom{n}{n-k} x^k \right] = \sum_{k=0}^{n} \left[\binom{n}{k}^2 \right] x^n$$

Hence, equating the coefficients of the terms for x^n $\qquad \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

$\Leftarrow$