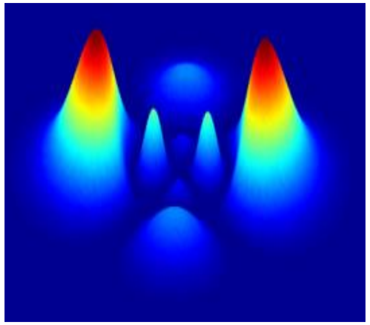

ADVANCED HIGH SCHOOL MATHEMATICS

INTEGRATION

METHODS OF INTEGRATION: INTEGRATION BY PARTS

One of the most useful and powerful integration **methods is integration by parts**.

Consider the differentiation of the function $y(x) = u(x)\, v(x)$

$$y(x) = u(x)\, v(x) \qquad y = u\, v \qquad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Now integrate this derivative with respect to x

$$\int \frac{dy}{dx}\, dx = \int u\frac{dv}{dx}\, dx + \int v\frac{du}{dx}\, dx$$

$$\int dy = \int d\left(uv\right) = u\, v = \int u\, dv + \int v\, du$$

$$\int u\, dv = u\, v - \int v\, du$$

This is often useful for transforming an integral that can't be integrated into one which may be integrable. **You have to be careful to make the correct choice for *u* and *v*.**

Example 1 $I = \int x \cos(x)\, dx$

$$I = \int x \cos(x)\, dx$$

$$u = x \qquad du = dx$$

$$dv = \cos(x)\, dx \qquad v = \sin(x)$$

$$\int u\, dv = u\, v - \int v\, du$$

$$I = \int x \cos(x)\, dx = x \sin(x) - \int \sin(x)\, du + C$$

$$I = x \sin(x) + \cos(x) + C$$

check the answer by differentiation

Example 2 $I = \int \log_e(x)\, dx$

$$I = \int \log_e(x)\, dx$$

$$u = \log_e(x) \qquad du = \left(\frac{1}{x}\right) dx$$

$$dv = dx \qquad v = x \qquad \int u\, dv = u\, v - \int v\, du$$

$$I = \int \log_e(x) dx = x \log_e(x) - \int x \left(\frac{1}{x}\right) dx + C$$

$$I = x \log_e(x) - x + C \qquad \log_e(x) \equiv \ln(x)$$

check the answer by differentiation

Example 3 $I=\int x\sqrt{4x+1}\,dx$

$$I=\int \cos^{-1}(x)\,dx$$

$$u=\cos^{-1}(x)\qquad du=\left(\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$dv=dx\qquad v=x$$

$$\int u\,dv=u\,v-\int v\,du$$

$$I=\int \cos^{-1}(x)\,dx=x\cos^{-1}(x)-\int x\left(\frac{1}{\sqrt{1-x^2}}\right)dx+C$$

$$\int x\left(\frac{1}{\sqrt{1-x^2}}\right)=\int x\left(1-x^2\right)^{-1/2}dx=-\left(1-x^2\right)^{1/2}$$

$$I=\int \cos^{-1}(x)\,dx=x\cos^{-1}(x)+\left(1-x^2\right)^{-1/2}+C$$

$$I=\int \cos^{-1}(x)\,dx=x\cos^{-1}(x)+\sqrt{1-x^2}+C$$

check the answer by differentiation

REDUCTION METHOD

It is often necessary to repeat the use of integration by parts to give **recurrence relationships**.

Example 4

$$I_n = \int x^n\, e^x\, dx$$

$$u = x^n \qquad du = n\, x^{n-1}\, dx$$

$$dv = e^x\, dx \qquad v = e^x$$

$$\int u\, dv = u\, v - \int v\, du$$

$$I_n = \int x^n\, e^x\, dx = x^n\, e^x - \int e^x\, n\, x^{n-1}\, dx$$

$$I_n = \int x^n\, e^x\, dx = x^n\, e^x - n \int x^{n-1} e^x\, dx$$

$$I_n = x^n\, e^x - n\, I_{n-1}$$

recurrence relationship or **reduction formula**

$$I_0 = \int x^0\, e^x\, dx = e^x$$

$$I_1 = x^1\, e^x - 1\, I_0 = x\, e^x - e^x$$

$$I_2 = x^2\, e^x - 2\, I_1 = x^2\, e^x - 2x\, e^x + 2e^x$$

etc

Example 5 $I_n = \int \sin^n(x)dx$

$$I_n = \int \sin^n(x)\,dx$$

$$u = \sin^{n-1}(x) \quad du = (n-1)\cos(x)\sin^{n-2}(x)\,dx$$

$$v\,dx = \sin(x) \quad v = -\cos(x)$$

$$I_n = \int \sin^{n-1}(x)\sin(x)dx = -\cos(x)\sin^{n-1}(x) - \int \big(-\cos(x)\big)(n-1)\cos(x)\sin^{n-2}(x)\,dx$$

$$I_n = -\cos(x)\sin^{n-1}(x) + (n-1)\int \cos^2(x)\,\sin^{n-2}(x)\,dx \qquad \cos^2(x) = 1 - \sin^2(x)$$

$$I_n = -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\,dx - (n-1)\int \sin^n(x)\,dx$$

$$I_n = -\cos(x)\sin^{n-1}(x) + (n-1)\,I_{n-2} - (n-1)I_n$$

$$I_n = \left(\frac{-1}{n}\right)\left(\cos(x)\sin^{n-1}(x)\right) + \left(\frac{n-1}{n}\right)I_{n-2}$$