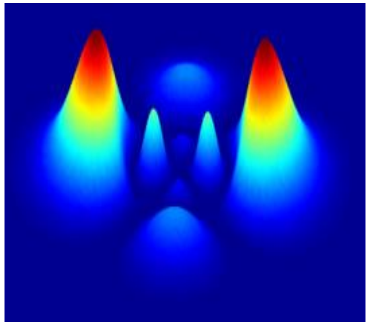

ADVANCED HIGH SCHOOL MATHEMATICS

INTEGRATION

METHODS OF INTEGRATION: Algebraic Manipulations

Integrals can often be evaluated by changing the integrand to a standard form by a suitable algebraic manipulation of the integrand. Remember **there are no general rules for integration and one has to rely on experience and trial and error**. Therefore, the best way to master this topic is by doing many integration exercises.

Example 1 $I = \int \tan(x)\, dx$

$$I = \int \tan(x)\, dx = \int \frac{\sin(x)\, dx}{\cos(x)}$$

$$u = \cos(x) \quad du = -\sin(x)\, dx \quad dx = \frac{-1}{\sin(x)}$$

$$I = -\int \frac{du}{u} = -\ln(u) + C$$

$$I = -\ln\left(\cos(x)\right) + C = \ln\left(\sec(x)\right) + C$$

check the answer by differentiation

Example 2 $I = \int \sin^2(x)\, dx$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x)$$
$$\sin^2(x) = \tfrac{1}{2}\left(1 - \cos(2x)\right)$$

$$I = \int \sin^2(x)\, dx = \tfrac{1}{2}\int \left(1 - \cos(2x)\right) dx$$
$$I = \left(\frac{x}{2} - \frac{\sin(2x)}{4}\right) + C$$

check the answer by differentiation

Example 3 $I = \int x\sqrt{4x+1}\, dx$

Integrals involving square toots may sometimes be simplified by the substitution of
$u = \sqrt{(\,)}$

$$u = (4x+1)^{1/2} \quad x = \tfrac{1}{4}\left(u^2 - 1\right) \quad dx = \tfrac{1}{2}u\, du$$
$$I = \int \tfrac{1}{4}\left(u^2 - 1\right) u\, \tfrac{1}{2}u\, du = \tfrac{1}{8}\int \left(u^4 - u^2\right) du$$
$$I = \tfrac{1}{40}u^5 - \tfrac{1}{24}u^3 + C$$
$$I = \tfrac{1}{40}(4x+1)^{5/2} - \tfrac{1}{24}(4x+1)^{3/2} + C$$

check the answer by differentiation

Example 4 $I = \int x^3 \sqrt{1-x^2}\, dx$

$$I = \int x^3 \left(1-x^2\right)^{1/2} dx$$

$$u = \left(1-x^2\right)^{1/2} \quad u^2 = \left(1-x^2\right) \quad x\,dx = -u\,du$$

$$I = \int x^2 \left(1-x^2\right)^{1/2} x\,dx = \int \left(1-u^2\right) u \left(-u\right) du$$

$$I = \int \left(u^4 - u^2\right) du$$

$$I = \tfrac{1}{5} u^5 - \tfrac{1}{3} u^3 + C$$

$$I = \tfrac{1}{5}\left(1-x^2\right)^{5/2} - \tfrac{1}{3}\left(1-x^2\right)^{3/2} + C$$

check the answer by differentiation

Example 5 $I = \int \frac{3x+1}{2x-3}\, dx$

Rational fractions: it is essential to express the fraction in such a way that the numerator is of a lower degree than the denominator.

$$I = \int \frac{3x+1}{2x-3}\, dx = \int \frac{\left(\frac{3}{2}\right)(2x-3) + \left(\frac{11}{2}\right)}{2x-3}\, dx$$

$$I = \int \left(\left(\tfrac{3}{2}\right) + \left(\tfrac{11}{2}\right)\left(\frac{1}{2x-3}\right) \right) dx$$

$$I = \left(\tfrac{3}{2}\right)x + \left(\tfrac{11}{2}\right)\left(\tfrac{1}{2}\right)\log_e(2x-3) + C$$

$$I = \left(\tfrac{3}{2}\right)x + \left(\tfrac{11}{4}\right)\log_e(2x-3) + C$$

check the answer by differentiation

Example 6 $I = \int \frac{x^2}{x+1}\, dx$

$$I = \int \frac{x^2}{x+1}\, dx = \int \left((x-1) + \frac{1}{x+1} \right) dx$$

$$I = \tfrac{1}{2}x^2 - x + \log_e(x+1) + C$$

check the answer by differentiation

Example 7 $I = \int \frac{3x+7}{2x^2+x-3}\,dx$

Use the fact that the quadratic denominator can be factorized into two separate factors, then write the equation as a sum of partial fractions.

$$I = \int \frac{3x+7}{2x^2+x-3}\,dx$$

$$2x^2+x-3=(2x+3)(x-1)$$

$$I = \int \frac{3x+7}{(2x+3)(x-1)}\,dx = \int\left(\frac{A}{2x+3}+\frac{B}{x-1}\right)dx$$

$$3x+7 = Ax - A + 2Bx + 3B \quad \Rightarrow \quad A+2B=3 \quad -A+3B=7 \quad \Rightarrow \quad B=2 \quad A=-1$$

$$I = \int\left(\frac{-1}{2x+3}+\frac{2}{x-1}\right)dx$$

$$I = \tfrac{-1}{2}\log_e(2x+3)+2\log_e(x-1)+C$$

check the answer by differentiation

Example 8 $I = \int \frac{1+x}{(1-x)^2}\, dx$

$$I = \int \frac{1+x}{(1-x)^2}\, dx$$

$$u = 1 - x \quad x = 1 - u \quad dx = -du$$

$$I = -\int \frac{2-u}{u^2}\, du = I = \int \left(\frac{1}{u} - \frac{2}{u^2} \right) du = \log_e(u) + 2u^{-1} + C$$

$$I = \log_e(1-x) + \frac{2}{1-x} + C$$

check the answer by differentiation

Example 9 $I = \int \frac{dx}{2x^2+3}$

standard integral $\int \frac{dx}{x^2+a^2} = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right)+C$

$$2x^2+3 = 2\left(x^2+3/2\right) \quad a=\sqrt{3/2}$$

$$I = (1/2)\int \frac{dx}{x^2+3/2} = \left(\sqrt{1/6}\right)\tan^{-1}\left(\sqrt{2/3}\;x\right)+C$$

check the answer by differentiation

If the denominator is of the form $ax^2+bx+c \quad b\neq 0$ then it can be reduced to the sum or difference of two squares by completing the sum of squares

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) = a\left\{(x+A)^2+B^2\right\}$$

$$x^2+\frac{b}{a}x+\frac{c}{a} = x^2+2A+A^2+B^2$$

$$A=\frac{b}{2a} \quad A^2+B^2=\frac{c}{a} \quad B^2=\frac{c}{a}-\frac{b^2}{4a^2}$$

Example 10 $I = \int \frac{dx}{x^2+4x+6}$

$$x^2+4x+6 = x^2+4x+2+2 = (x+2)^2+2$$

$$I = \int \frac{dx}{x^2+a^2} = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right)+C$$

$$I = \int \frac{dx}{(x+2)^2+2} \quad a=\sqrt{2}$$

$$I = \left(\sqrt{1/2}\right)\tan^{-1}\left(\frac{x+2}{\sqrt{1/2}}\right)+C$$

check the answer by differentiation

Example 11 $I = \int \frac{dx}{x^2 + 6x - 7}$

$$x^2 + 6x - 7 = x^2 + 6x + 9 - 16 = (x+3)^2 - 16 = (x+3-4)(x+3+4)$$

$$x^2 + 6x - 7 = (x-1)(x+7)$$

$$\frac{1}{x^2 + 6x - 7} = \frac{1}{(x-1)(x+7)} = \frac{A}{x-1} + \frac{B}{x+7}$$

$$1 = Ax + 7A + Bx - B \quad A + B = 0 \quad 7A - B = 1 \quad A = \frac{1}{8} \quad B = -\frac{1}{8}$$

$$I = \frac{1}{8}\int \frac{dx}{x-1} - \frac{1}{8}\int \frac{dx}{x+7} = \left(\frac{1}{8}\right)\left(\log_e(x-1) - \log_e(x+7)\right) + C$$

$$I = \left(\frac{1}{8}\right)\log_e\left(\frac{x-1}{x+7}\right) + C$$

check the answer by differentiation

Example 12 $I=\int \frac{(4x+5)\,dx}{3x^2+x+3}$

$$I=\int \frac{(4x+5)\,dx}{3x^2+x+3}$$

$$y=3x^2+x+3 \quad dy/dx=6x+1$$

$$\left(\frac{2}{3}\right)(6x+1)=4x+\frac{2}{3}$$

$$\left(\frac{2}{3}\right)(6x+1)+\frac{13}{3}=4x+\frac{2}{3}+\frac{13}{3}=4x+5$$

$$I=\left(\frac{2}{3}\right)\int \frac{(6x+1)\,dx}{3x^2+x+3}+\left(\frac{13}{3}\right)\int \frac{dx}{3x^2+x+3}$$

$$I_1=\left(\frac{2}{3}\right)\log_e\left(3x^2+x+3\right)+C_1 \quad I_2=\left(\frac{13}{3}\right)\int \frac{dx}{3x^2+x+3}$$

$$3x^2+x+3=3\left(x^2+x/3+1\right)=3\left(x^2+x/3+1/36+35/36\right)=3\left((x+1/6)^2+35/36\right)$$

$$I_2=\left(\frac{13}{3}\right)\int \frac{dx}{3x^2+x+3}=\left(\frac{13}{9}\right)\int \frac{dx}{(x+1/6)^2+35/36}$$

$$I_2=\left(\frac{13}{9}\right)\left(\sqrt{\frac{36}{35}}\right)\tan^{-1}\left(\sqrt{\frac{36}{35}}(x+1/6)\,x\right)+C_2$$

$$I_2=\left(\frac{26}{3\sqrt{35}}\right)\tan^{-1}\left(\frac{6x+1}{\sqrt{35}}x\right)+C_2$$

$$I=I_1+I_2$$

$$I=\left(\frac{2}{3}\right)\log_e\left(3x^2+x+3\right)+\left(\frac{26}{3\sqrt{35}}\right)\tan^{-1}\left(\frac{6x+1}{\sqrt{35}}x\right)+C$$

check the answer by differentiation

Example 13 $I = \int \frac{dx}{\sqrt{4-2x-x^2}}$

$$(1+x)^2 = 1+2x+x^2 \quad 4-2x-x^2 = 5-\left(1+2x+x^2\right) = 5-(1+x)^2$$

$$I = \int \frac{dx}{\sqrt{5-(1+x)^2}}$$

Standard integral $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C = -\cos^{-1}\left(\frac{x}{a}\right) + C \quad -a < x < a \quad a > 0$

$$I = \sin^{-1}\left(\frac{1+x}{\sqrt{5}}\right) + C$$

check the answer by differentiation

Example 14 $I = \int \frac{(2x+3)\,dx}{\sqrt{1-x-x^2}}$

$$I = \int \frac{(2x+3)\,dx}{\sqrt{1-x-x^2}}$$

$$y = 1-x-x^2 \quad dy/dx = -1-2x$$

$$2x+1 = -dy/dx \quad 2x+3 = -dy/dx + 2$$

$$I = -\int \frac{(-1-2x)\,dx}{\sqrt{1-x-x^2}} + \int \frac{2\,dx}{\sqrt{1-x-x^2}}$$

$$I_1 = -\int \frac{(-1-2x)\,dx}{\sqrt{1-x-x^2}} = -\int (-1-2x)\left(1-x-x^2\right)^{-1/2} dx$$

$$I_1 = -2\left(1-x-x^2\right)^{1/2} + C_1 = -2\sqrt{1-x-x^2} + C$$

$$I_2 = 2\int \frac{dx}{\sqrt{1-x-x^2}}$$

$$(1/2+x)^2 = 1/4 + x + x^2 \quad 5/4 - (1/2+x)^2 = 5/4 - -1/4 - x - x^2 = 1-x-x^2$$

$$I_2 = 2\int \frac{dx}{\sqrt{5/4-(1/2+x)^2}}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C = -\cos^{-1}\left(\frac{x}{a}\right) + C \quad -a < x < a \quad a > 0$$

$$I_2 = 2\sin^{-1}\left(\frac{1+2x}{\sqrt{5}}\right) + C_2$$

$$I = I_1 + I_2$$

$$I = I_1 = -2\left(1-x-x^2\right)^{1/2} + C_1 = -2\sqrt{1-x-x^2} + 2\sin^{-1}\left(\frac{1+2x}{\sqrt{5}}\right) + C$$

check the answer by differentiation

Example 15 $I = \int \frac{(2x+3)\,dx}{x\sqrt{x^2 - x - 1}}$

$$I = \int \frac{(2x+3)\,dx}{x\sqrt{x^2 - x - 1}}$$

$$u = \frac{1}{x} \quad x = \frac{1}{u} \quad dx = -\frac{du}{u^2}$$

$$I = -\int \frac{du}{u^2(1/u)\sqrt{(1/u^2) - (1/u) - 1}}$$

$$I = -\int \frac{du}{\sqrt{1 - u - u^2}}$$

$$(1/2 + u)^2 = 1/4 + u + u^2 \quad 5/4 - (1/2 + u)^2 = 1 - u - u^2$$

$$I = -\int \frac{du}{\sqrt{5/4 - (1/2 + u)^2}}$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C = -\cos^{-1}\left(\frac{x}{a}\right) + C \quad -a < x < a \quad a > 0$$

$$I = \cos^{-1}\left(\frac{2(1/2 + u)}{\sqrt{5}}\right) + C$$

$$I = \cos^{-1}\left(\frac{x+2}{x\sqrt{5}}\right) + C$$

check the answer by differentiation