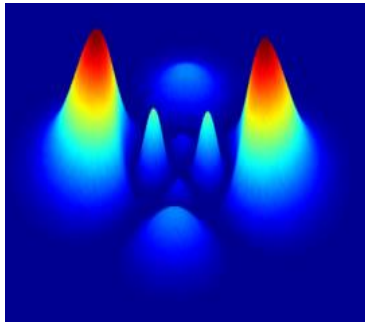


ADVANCED HIGH SCHOOL MATHEMATICS

INTEGRATION

METHODS OF INTEGRATION: SUBSTITUTION

Integrals can often be evaluated by changing the integrand to a standard form by a suitable substitution in the variable. **Unfortunately, there are no general rules for making substitutions and one has to rely on experience and trial and error**. Therefore, the best way to master this topic is by doing many integration exercises.

Example 1 $I = \int \frac{dx}{\sqrt{a\,x + b}}$

Substitution $u = a\,x + b \quad du/dx = a \quad dx = (1/a)\,du$

$$I = \left(\frac{1}{a}\right)\int \frac{du}{\sqrt{u}} = \left(\frac{1}{a}\right)\int u^{-1/2}\,du = \left(\frac{2}{a}\right)u^{1/2} + C$$

$$I = \left(\frac{2}{a}\right)\sqrt{a\,x + b} + C$$

Always check the answer by differentiation $dI / dx = \frac{1}{\sqrt{a\,x + b}}$

Example 2 $I = \int \cos(5y + 99)\, dy$

Substitution $u = 5y + 99 \quad du/dy = 5 \quad dy = (1/5)\, du$

$$I = (1/5)\int \cos(u)\, du = (1/5)\sin(u) + C$$
$$I = (1/5)\sin(5y + 99) + C$$

Always check the answer by differentiation $dI/dy = \cos(5y + 99)$

Example 3 $I = \int \frac{dt}{a + bt}$

Substitution $u = a + bt \quad du/dt = b \quad dt = (1/b)\, du$

$$I = (1/b)\int \frac{du}{u} = (1/b)\ln(u)$$
$$I = (1/b)\ln(a + bt)$$

Always check the answer by differentiation $dI/dt = \frac{1}{a + bt}$

Integrals of the form

$$I = \int [\, f(x)]^n \; f'(x)\, dx$$

can be integrated by the substitution

$$u = f(x) \quad du = f'(x)\, dx$$

Example 4 $I = \int \frac{3x}{1+6x^2}\, dx$

Substitution $u = 1 + 6x^2 \quad du/dx = 12x \quad dx = (1/12\,x)\, du$

$$I = (1/4)\int \frac{du}{u} = (1/4)\ln(u)$$

$$I = (1/4)\ln(1+6x^2)$$

Always check the answer by differentiation $dI/dx = \frac{3}{1+6x^2}$

Example 5 $I = \int \left(x^3 + 2x^2 + 5x + 7\right)^{11} \left(6x^2 + 8x + 10\right) dx$

Substitution $u = x^3 + 2x^2 + 5x + 7 \quad du/dx = 3x^2 + 4x + 5 \quad dx = \dfrac{du}{3x^2 + 4x + 5}$

$$I = 2\int u^{11}\, du = \left(2/12\right) u^{12} + C$$

$$I = \left(1/6\right)\left(x^3 + 2x^2 + 5x + 7\right)^{12} + C$$

Always check the answer by differentiation $dI/dx = \left(x^3 + 2x^2 + 5x + 7\right)^{11}\left(6x^2 + 8x + 10\right)$

Example 6 $I = \int \dfrac{dx}{\left(a - x\right)^2}$

Substitution $u = a - x \quad du/dx = -1 \quad dx = -\,du$

$$I = -\int \frac{du}{u^2} = -\int u^{-2}\, du = u^{-1} + C$$

$$I = \frac{1}{a - x} + C$$

Example 7 $I = \int x \sqrt{1+5x}\, dx$

Substitution
$$u = \sqrt{1+5x} = (1+5x)^{1/2} \quad x = (u^2 - 1)/5$$
$$du/dx = (5/2)(1+5x)^{-1/2} \quad dx = (2/5)(1+5x)^{1/2}\, du$$

$$I = (1/5)(2/5)\int (u^2 - 1)\, u\, u\, du = \left(\frac{2}{25}\right)\int (u^4 - u^2)\, du$$

$$I = \left(\frac{2}{25}\right)\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C = \frac{2u^5}{125} - \frac{2u^3}{75} + C$$

$$I = \left(\frac{2}{125}\right)(1+5x)^{5/2} - \left(\frac{2}{75}\right)(1+5x)^{3/2} + C$$

Always check the answer by differentiation $dI/dx = x\sqrt{1+5x}$

Example 8 $I = \int x^3 \sqrt{1-x^2}\, dx$

$u = \sqrt{1-x^2} = \left(1-x^2\right)^{1/2} \qquad x = \left(1-u^2\right)^{1/2}$

Substitution

$du/dx = -x\,(1-x^2)^{-1/2} \qquad dx = \frac{-\left(1-x^2\right)^{1/2} du}{x} = \frac{-u\, du}{\left(1-u^2\right)^{1/2}}$

$$I = -\int \left(1-u^2\right)^{3/2} u \;\frac{u\, du}{\left(1-u^2\right)^{1/2}} = -\int \left(1-u^2\right) u^2 du$$

$$I = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$I = (1/5)\left(1-x^2\right)^{5/2} - (1/3)\left(1-x^2\right)^{3/2} + C$$

Always check the answer by differentiation $dI/dx = x^3 \sqrt{1-x^2}$

Example 9 $I = \int \frac{dx}{2+\sqrt{x}}$

Substitution $u = \sqrt{x} = x^{1/2} \quad x = u^2$

$dx = 2u\,du$

$$I = \int \frac{2u\,du}{u+2} = 2\int \frac{u+2-2}{u+2}du = 2\left[\int \frac{u+2}{u+2}du + \int \frac{-2}{u+2}du\right]$$

$$I = 2\left[u - 2\ln\left(u+2\right)\right] + C$$

$$I = 2\sqrt{x} - 4\ln\left(\sqrt{x}+2\right) + C$$

Always check the answer by differentiation $dI/dx = \frac{1}{2+x^{1/2}}$

Trigonometry Review

Often the substitution of a trigonometric function reduces the integral to standard form.

Example 10 $I = \int \frac{dx}{\sqrt{a^2 - x^2}}$

$$x = a\sin(u) \quad dx = -a\cos(u)\,du$$

$$I = \int \frac{\cos(u)\,du}{\sqrt{1-\sin^2(u)}} = u + C$$

$$I = \sin^{-1}\left(\frac{x}{a}\right) + C$$

Example 11 Find the area A of a semicircle of radius a

Equation of a circle of radius a $x^2 + y^2 = a^2$

Equation of semicircle $y = \sqrt{a^2 - x^2} \quad -a \le x \le +a$

Area of semicircle $A = \int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$

$$x = a\sin(u) \quad dx = a\cos(u)\, du$$
$$x = -a \to u = -\pi/2$$
$$x = a \to u = \pi/2$$
$$A = \int_{-\pi/2}^{\pi/2} \sqrt{a^2 - a^2 \sin^2(u)}\; a\cos(u)\, du$$
$$A = a^2 \int_{-\pi/2}^{\pi/2} \cos^2(u)\, du$$
$$\cos(2u) = \cos^2(u) - \sin^2(u)$$
$$\cos^2(u) + \sin^2(u) = 1$$
$$\cos^2(u) = (1/2)\left(1 + \cos(2u)\right)$$
$$A = \left(\frac{a^2}{2}\right) \int_{-\pi/2}^{\pi/2} \left[1 + \cos(2u)\right] du$$
$$A = \left(\frac{a^2}{2}\right) \left[u + \tfrac{1}{2}\sin(2u)\right]_{-\pi/2}^{\pi/2}$$
$$A = \left(\frac{a^2}{2}\right) \left[\pi + 0\right]$$
$$A = \frac{\pi a^2}{2}$$

If the integrand involves terms such as $\sqrt{1 + x^2}$ then the substitution $u = \tan(x)$ may be useful.

Example 12 $I = \int \frac{dx}{x^2\sqrt{1+x^2}}$

$$x = \tan(u) \quad dx = \sec^2(u)\,du$$

$$\sec^2(u) = 1 + \tan^2(u) \quad \tan(u) = \frac{\sin(u)}{\cos(u)}$$

$$I = \int \left[\frac{\sec^2(u)}{\tan^2(u)\sqrt{1+\tan^2(u)}}\right] du$$

$$I = \int \left[\frac{\sec(u)}{\tan^2(u)}\right] du$$

$$I = \int \left[\frac{\cos(u)}{\sin^2(u)}\right] du$$

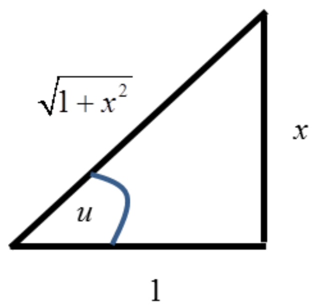
$$I = \frac{-1}{\sin(u)} + C$$

$$\sin(u) = \frac{x}{\sqrt{1+x^2}}$$

$$I = \frac{-\sqrt{1+x^2}}{x} + C$$

$$\tan(u) = \frac{x}{1}$$

$$\sin(u) = \frac{x}{\sqrt{1+x^2}}$$



Some useful trigonometric relations needed for evaluating many types of integrals

$$\sin^2(x)+\cos^2(x)=1 \quad \sin(x)=\sqrt{1-\cos^2(x)} \quad \cos(x)=\sqrt{1-\sin^2(x)}$$

$$\tan^2(x)+1=\frac{\sin^2(x)}{\cos^2(x)}+1=\frac{\sin^2(x)+\cos^2(x)}{\cos^2(x)}=\frac{1}{\cos^2(x)}$$

$$\sec^2(x)=\tan^2(x)+1$$

$$\cos(2x)=\cos^2(x)-\sin^2(x)=2\cos^2(x)-1 \quad \cos^2(x)=\tfrac{1}{2}\left(1+\cos(2x)\right)$$

$$\sin(2x)=2\sin(x)\cos(x)$$

$$\tan(2x)=\frac{\sin(2x)}{\cos(2x)}=\frac{2\sin(x)\cos(x)}{\cos^2(x)-\sin^2(x)} \qquad \div\frac{\cos^2(x)}{\cos^2(x)}$$

$$\tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)}$$

$$\cos(2x)=\cos^2(x)-\sin^2(x)=\cos^2(x)\left(1-\frac{\sin^2(x)}{\cos^2(x)}\right)=\left(\frac{1}{\sec^2(x)}\right)\left(1-\tan^2(x)\right)$$

$$\cos(2x)=\frac{1-\tan^2(x)}{1+\tan^2(x)}$$

$$\sin(2x)=2\sin(x)\cos(x)=\frac{2\sin(x)\cos^2(x)}{\cos(x)}=\frac{2\tan x}{\sec^2(x)}$$

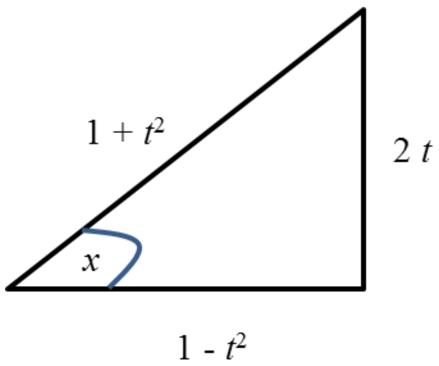
$$\sin(2x)=\frac{2\tan x}{1+\tan^2(x)}$$

The substitution $t = \tan(x/2)$ is often a useful one for integration of trigonometric functions because we can express

$$\sin(x) = \frac{2t}{1+t^2} \quad \cos(x) = \frac{1-t^2}{1+t^2} \quad dx = \frac{2\,dt}{1+t^2}$$

$$t = \tan\left(x/2\right) \quad dt = \frac{1}{2}\left(1+\tan^2(x/2)\right)dx = \frac{1}{2}\left(1+t^2\right)dx \quad dx = \frac{2}{1+t^2}dt$$

$$\cos(x) = \frac{1-t^2}{1+t^2} \quad \cos(x)1+t^2 = 1-t^2 \Rightarrow t^2 = \frac{1-\cos(x)}{1+\cos(x)}$$



$$t = \tan(x/2)$$

$$\sin x = \frac{2t}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2} \quad \tan x = \frac{2t}{1-t^2}$$

$$dx = \frac{2}{1+t^2}\,dt \quad t^2 = \frac{1-\cos x}{1+\cos x}$$

Example 13 $I = \int \operatorname{cosec}(x)\, dx$

$$I = \int \operatorname{cosec}(x)\, dx = \int \frac{dx}{\sin(x)} = \int \left(\frac{1+t^2}{2t} \right) \frac{2\, dt}{1+t^2} = \int \frac{dt}{t}$$

$$I = \ln(t) + C = \ln\left(\tan\left(\frac{x}{2} \right) \right) + C$$

Example 14 $I = \int \frac{d\theta}{2+\sin\theta}$

$$I = \int \frac{d\theta}{2+\sin\theta} \qquad \sin\theta = \frac{2t}{1+t^2} \qquad d\theta = \frac{2}{1+t^2}\,dt$$

$$\frac{d\theta}{2+\sin\theta} = \left(\frac{2}{1+t^2}\right)\left(\frac{1}{2+\frac{2t}{1+t^2}}\right)dt = \frac{dt}{t^2+t+1}$$

$$t^2+t+1 = (t+A)^2 + B^2 = t^2 + 2At + A^2 + B^2$$

$$A = 1/2 \qquad B^2 = 3/4 \qquad B = \sqrt{3}/2$$

$$\frac{d\theta}{2+\sin\theta} = \frac{dt}{(t+A)^2+B^2}$$

substitute numerical values for A and B later is best

$$I = \int \frac{dt}{(t+A)^2+B^2} \qquad \int \frac{dx}{a^2+x^2} = \left(\frac{1}{a}\right)\tan^{-1}\left(\frac{x}{a}\right)$$

$$I = \frac{1}{B}\tan^{-1}\left(\frac{t+A}{B}\right) + K$$

$$I = \left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\left(\frac{2}{\sqrt{3}}\right)\left(\tan(\theta/2)+1/2\right)\right) + K$$

$$I = \left(\frac{2\sqrt{3}}{3}\right)\tan^{-1}\left(\left(\frac{2\sqrt{3}}{3}\right)\left(\tan(\theta/2)+1/2\right)\right) + K$$