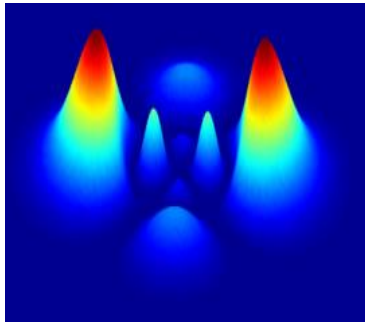


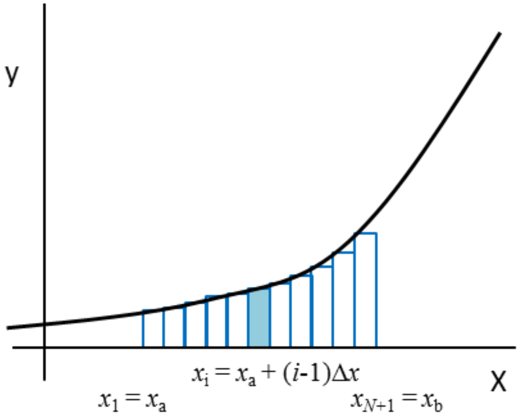
ADVANCED HIGH SCHOOL MATHEMATICS

INTEGRATION

DEFINITE INTEGRALS

Integration is a very valuable technique for calculations that can be expressed in terms of a summation. An important theorem relates the limit of a summation to the definite integral. Consider the continuous and single-valued function $f(x)$ defined in the interval $x_a \le x \le x_b$. The interval from x_a to x_b can be divided into N equal subintervals each of length Δx where $\Delta x = \dfrac{x_b - x_a}{N}$ and

$$x_1 = x_a \quad x_2 = x_a + \Delta x \quad x_3 = x_a + 2\Delta x \quad \cdots \quad x_{N+1} = x_b$$



Then the sum of the rectangles S_N corresponds to

$$S_N = \sum_{i=1}^{N} f\left(x_i\right) \Delta x$$

The **fundamental theorem of integral calculus** states that as the number of subintervals N approaches infinity

$$\underset{N\to\infty}{\text{limit}}\, S_N = \underset{N\to\infty}{\text{limit}} \sum_{i=1}^{N} f\left(x_i\right) \Delta x = \int_{x_a}^{x_b} f(x)\, dx$$ **definite integral**

The definite integral corresponds to the area A under the curve represented by the function $y = f(x)$ in the interval $x_a \le x \le x_b$.

$$A = \int_{x_a}^{x_b} f(x)\, dx$$

where x_a is the **lower limit** of the integration and x_b is the **upper limit**. The function $f(x)$ is called the **integrand** and A is called the **integral**.

The definite integral can be expressed as: if there is some function $F(x)$ which is differentiable in the interval $x_a \le x \le x_b$ and has the derivative $f(x)$ then the definite integral of $f(x)$ with respect to x over the interval is

$$F(x_b) - F(x_a) = \int_{x_a}^{x_b} f(x)\, dx$$ **definite integral**

Example

Evaluate $\int_1^2 \left(x^2 + 2x + 1\right) dx$

indefinite integral

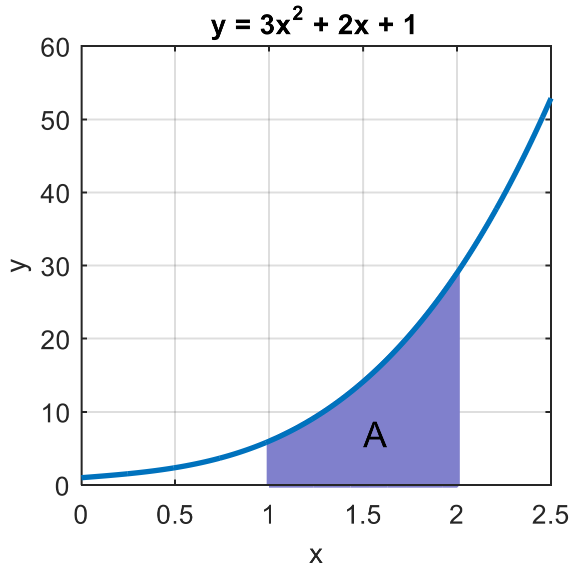
$$\int_1^2 \left(3x^2 + 2x + 1\right) dx$$

$$F(x) = \int \left(3x^2 + 2x + 1\right) dx = x^3 + x^2 + x + C$$

$$F(2) = 8 + 4 + 2 + C \quad F(1) = 1 + 1 + 1 + C$$

$$F(2) - F(1) = \left(8 + 4 + 2 + C\right) - \left(1 + 1 + 1 + C\right)$$

$$F(2) - F(1) = 11$$



definite integral

$$A = \int_1^2 \left(3x^2 + 2x + 1\right) dx$$

$$A = \left[x^3 + x^2 + x\right]_1^2$$

$$A = \left(2^3 - 1^3\right) + \left(2^2 - 1^2\right) + \left(2 - 1\right)$$

$$A = 7 + 3 + 1 = 11$$

Properties of definite integrals

- Interchanging the limits of integration changes the sign of the integral

$$\int_{x_a}^{x_b} f(x)\,dx = -\int_{x_b}^{x_a} f(x)\,dx$$

- The range of integration can be subdivided

$$\int_{x_a}^{x_b} f(x)\,dx = \int_{x_a}^{x_c} f(x)\,dx + \int_{x_c}^{x_b} f(x)\,dx \qquad x_a < x_c < x_b$$

- Integration by substitution $x = h(u)$ Need to change the limits of integration

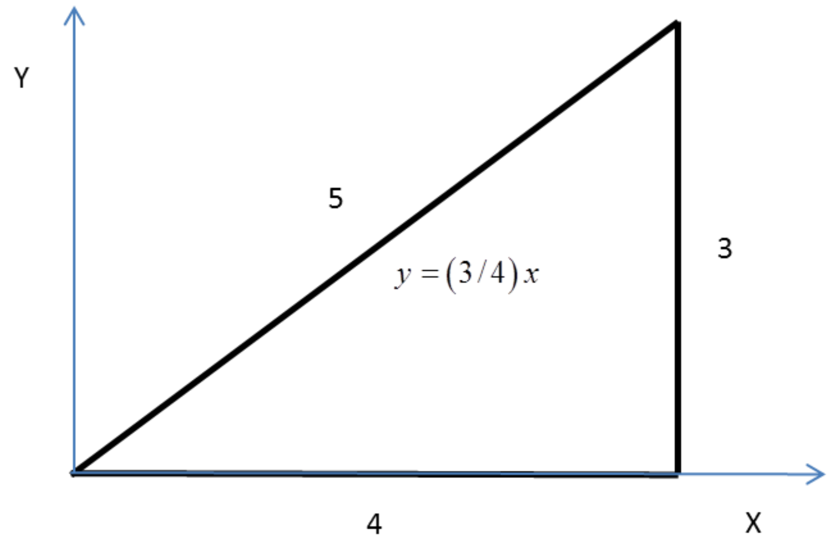
$$\int_{x_a}^{x_b} f(x)\,dx = \int_{u_b}^{u_a} g(u)\,du$$

- Integration of even and odd functions

even function $f(-x) = f(x)$ $\quad \int_{-x_a}^{x_a} f(x)\,dx = 2\int_{0}^{x_a} f(x)\,dx$

odd function $f(-x) = -f(x)$ $\quad \int_{-x_a}^{x_a} f(x)\,dx = 0$

Example Find the area of the triangle with sides 3, 4 and 5.



Equation of hypotenuse $y = (3/4)x \quad 0 \le x \le 4$

Area of triangle equals area under curve $A = \int_{x_a}^{x_b} y \, dx$

$$A = \int_0^4 (3/4)\, x \, dx = (3/8)\left[x^2\right]_0^4$$

$$A = 6 \qquad A = \left(\tfrac{1}{2}\right)(base)(height) = \left(\tfrac{1}{2}\right)(4)(3) = 6$$