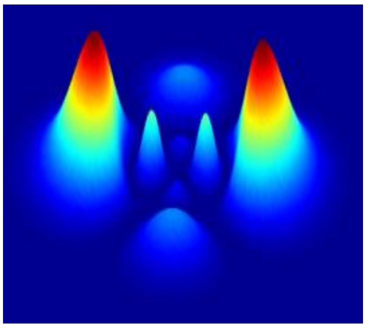

ADVANCED HIGH SCHOOL MATHEMATICS

INTEGRATION

INDEFINITE INTEGRALS

Differentiation is concerned with the rates of change of physical quantities. It is a fundamental topic in mathematics, physics, chemistry, engineering etc. Consider a continuous and single value function $y = f(x)$. The rate of change of y with respect to x at the point x_1 is called the **derivative** and equals the slope of the tangent to the curve $y = f(x)$ at the point x_1. The process of finding the derivative of a function is called **differentiation**.

The reverse problem is also an extremely important part of mathematics – given a function $f(x)$ which represents the rate of change of some quantity with respect to x, what is the function $F(x)$ such that

$$f(x) = \frac{d\,F(x)}{dx}$$

The process of working back from the derivative of $F(x)$ to the function $F(x)$ is called **integration**.

$$F(x) = \int f(x)\, dx$$ **indefinite integral**

Integration is the inverse process of differentiation. Whereas there are definite rules for the differentiation of any function, **there are no such rules for integration**.

Given that

$$f(x) = \frac{d\,F(x)}{dx}$$

then $F(x)$ called an **indefinite integral** because the value $F(x)$ depends upon some arbitrary constant C.

$$F(x) = \int f(x)\, dx + C$$ C is a constant

Example

$$y = 3x^2 + 2x + 15$$
$$z = 3x^2 + 2x + 99$$
$$dy/dx = 6x + 2 \quad dz/dx = 6x + 2$$
$$y \neq z \quad dy/dx = dz/dx$$
$$F(x) = \int (dy/dx)\,dx = \int (dz/dx)\,dx = \int (6x+2)\,dx$$
$$F(x) = 3x^2 + 2x + C$$

The value of C can't be determined from dy/dx or dz/dx alone.

The value of an integral is unaffected by multiplication by a constant

$$\int a\, f(x)\, dx = a \int f(x)\, dx \qquad a \text{ is a constant}$$

The integral of a sum can be found by integrating each term separately

$$\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$$

STANDARD INTEGRALS

Some integrals are of a standard form and can be integrated relatively easily and others can be converted to a standard form using a variety of different techniques. Many integrals can't be integrated analytically at all, for example, $\int \exp(-x^2)\, dx$. However, the integral can be evaluated using numerical techniques such as Simpson's rule.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1 \quad x \neq 0$$

$$\int \frac{1}{x} \, dx = \int x^{-1} \, dx = \log_e(x) + C \quad \log_e(x) \equiv \ln(x) \quad x > 0$$

$$\int \sin(a\,x) \, dx = -\frac{1}{a}\cos(a\,x) + C \quad a \neq 0$$

$$\int \cos(a\,x) \, dx = \frac{1}{a}\sin(a\,x) + C \quad a \neq 0$$

$$\int \sec^2(a\,x) \, dx = \frac{1}{a}\tan(a\,x) + C \quad a \neq 0$$

$$\int \operatorname{cosec}^2(a\,x) \, dx = -\frac{1}{a}\cot(a\,x) + C \quad a \neq 0$$

$$\int \sec(a\,x)\tan(a\,x) \, dx = \frac{1}{a}\sec(a\,x) + C \quad a \neq 0$$

$$\int \operatorname{cosec}(a\,x)\cot(a\,x) \, dx = -\frac{1}{a}\operatorname{cosec}(a\,x) + C \quad a \neq 0$$

$$\int e^{a\,x} \, dx = \frac{1}{a} e^{a\,x} + C \quad a \neq 0$$

$$\int a^{x} \, dx = \frac{a^{x}}{\log e(a)} + C \quad a \neq 1 \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + C = -\cos^{-1}\left(\frac{x}{a}\right) + C \quad -a < x < a \quad a > 0$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \log_e\left(x + \sqrt{x^2 - a^2}\right) + C \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \log_e\left(x + \sqrt{x^2 + a^2}\right) + C$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \quad a \neq 0$$