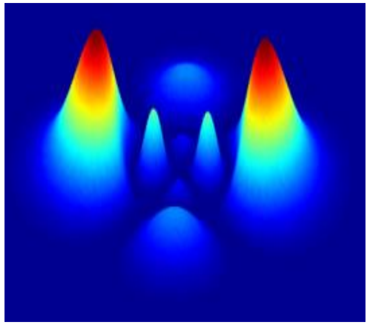


# ADVANCED HIGH SCHOOL MATHEMATICS

## GRAPH THEORY

## Problem solving using graphs and inequality

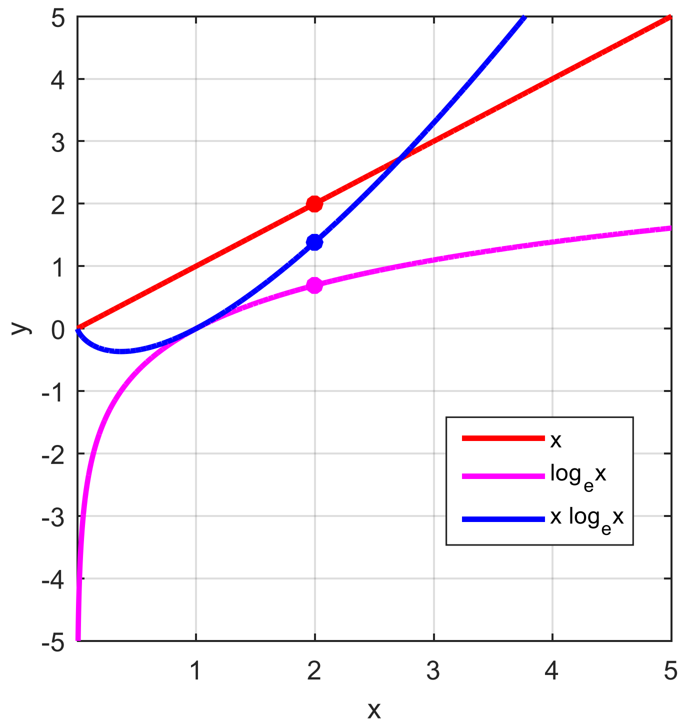
### OPERATING ON GRAPHS OF BASIC FUNCTIONS

If you want to graph the function $y = x\log_e(x)$ how do you start?

Step 1. Plot the graph of $y_1 = x$ and then $y_2 = \log_e(x)$ and consider the properties of each graph.

Step 2. Plot the graph of $y = x\log_e(x) = y_1\, y_2$ by considering that at each point $x_1$ that $y(x_1) = x_1 \log_e(x_1)$

$$x_1 = 2$$
$$y_1 = x_1 = 2$$
$$y_2 = \log_e(x_1) = \log_e(2) = 0.6931$$
$$y = y_1 y_2 = (2)(0.6931) = 1.3862$$




Note: line $y = x$ is not at 45° because the X-axis and Y-axis have different scales

Graphing a function: The **domain** is the set of all first elements of ordered pairs (X-coordinates) and the **range** is the set of all second elements of ordered pairs (Y-coordinates).

Graph the function $y = 2\sin(3x) + x$ in the domain $-\pi \le x \le \pi$

How to approach the problem:

Find the zeros, max and min for $\sin(\theta)$ and for $\sin(3x)$ $\quad -\pi \le x \le \pi \quad -2 \le y \le +2$

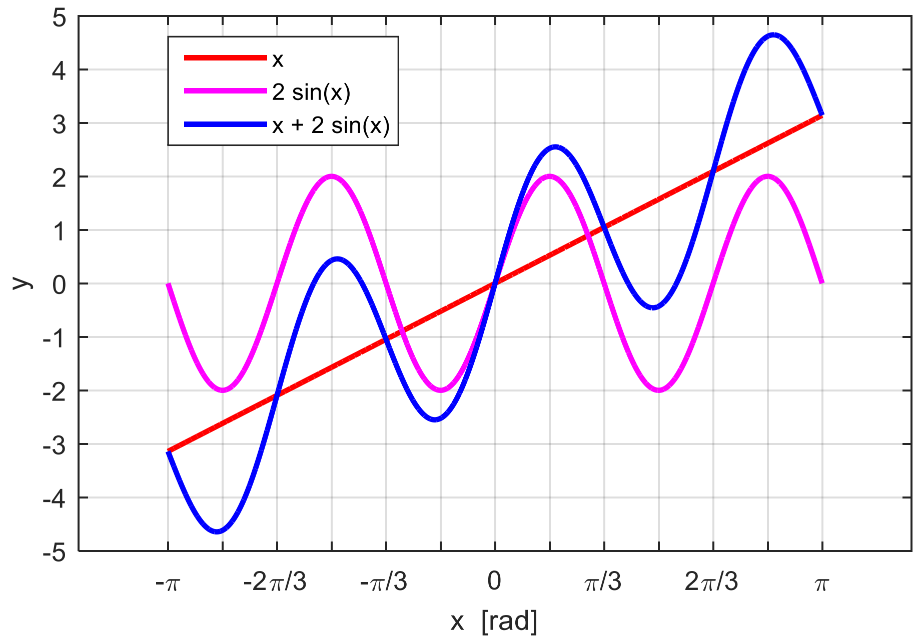
Graph $y_1 = 2\sin(3x) \quad y_2 = x \quad \Rightarrow \quad y = y_1 + y_2$

Answer:

$$\sin(\theta) = \sin(3x) = 0 \quad \Rightarrow \quad \theta = 3x = 0, \pm\pi, \pm 2\pi, \pm 3\pi \quad \Rightarrow \quad x = 0, \pm\pi/3, \pm 2\pi/3, \pm\pi$$

$$\sin(\theta) = \sin(3x) = 1 \quad \Rightarrow \quad \theta = 3x = \pi/2, \pi/2 \pm 2\pi \quad \Rightarrow \quad x = \pi/6, 5\pi/6, -\pi/2$$

$$\sin(\theta) = \sin(3x) = -1 \quad \Rightarrow \quad \theta = 3x = -\pi/2, -\pi/2 \pm 2\pi \quad \Rightarrow \quad x = -\pi/6, -5\pi/6, \pi/2$$



Graph the function $y = 2\sin(3x-3) + x$ in the domain $-\pi \le x \le \pi$

How to approach the problem:

Find the zeros, max and min for $\sin(\theta)$ and for $\sin(3x-3)$ $\quad -\pi \le x \le \pi \quad -2 \le y \le +2$

Graph $y_1 = 2\sin(3x-3) \quad y_2 = x \quad \Rightarrow \quad y = y_1 + y_2$

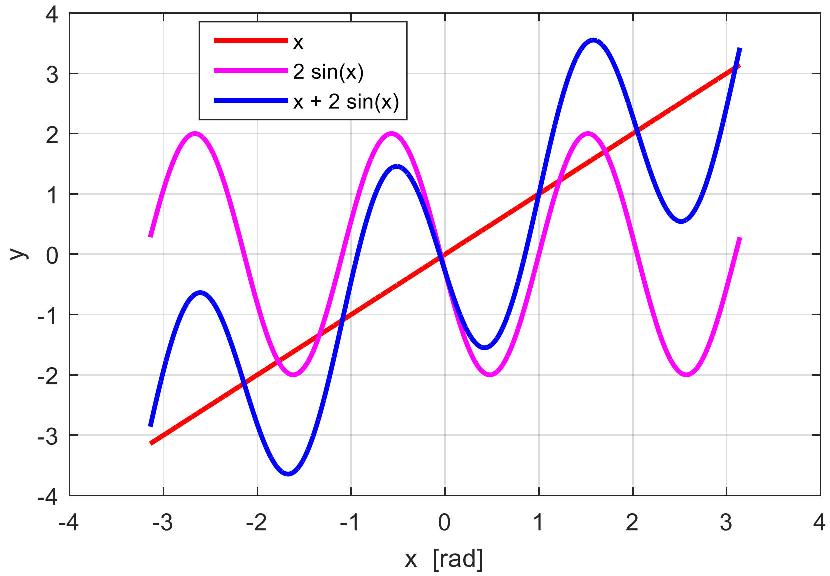
Answer:

$$\sin(\theta) = \sin(3x-3) = 0 \quad \Rightarrow \quad \theta = 3x - 3 = 0, \pi, 2\pi, 3\pi$$
$$\Rightarrow \quad x = 1, 1+\pi/3, 1+2\pi/3 \quad \Rightarrow \quad x = 1, 2.0472, 3.0944$$

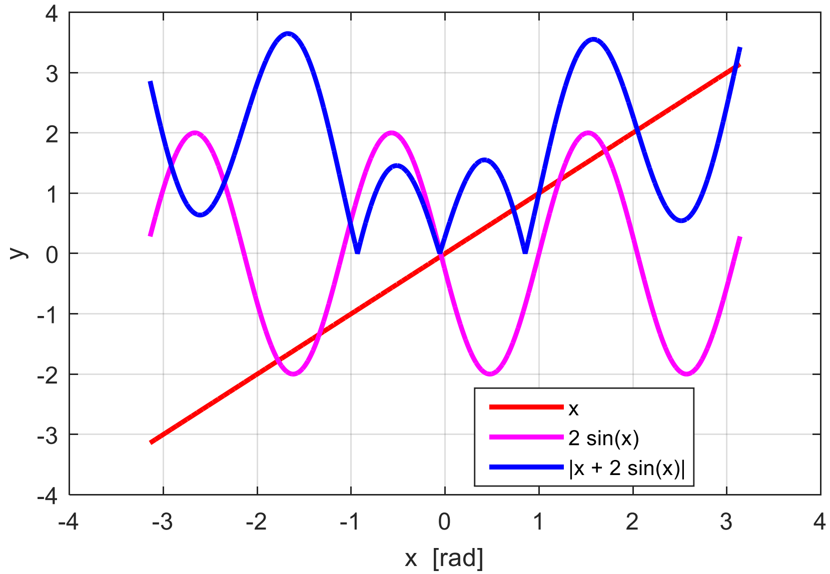
$$\sin(\theta) = \sin(3x-3) = 0 \quad \Rightarrow \quad \theta = 3x - 3 = -\pi, -2\pi, -3\pi$$
$$\Rightarrow \quad x = 1-\pi/3,\ 1-2\pi/3, 1-\pi \quad \Rightarrow \quad x = -0.1472, -1.0944, -2.1416$$

$$\sin(\theta) = \sin(3x-3) = 1 \quad \Rightarrow \quad \theta = 3x - 3 = \pi/2 \quad \Rightarrow \quad x = 1+\pi/6 = 1.5236$$
$$\sin(\theta) = \sin(3x-3) = 1 \quad \Rightarrow \quad \theta = 3x - 3 = -3\pi/2, -7\pi/2$$
$$\Rightarrow \quad x = 1-3\pi/6, 1-7\pi/6 \quad \Rightarrow \quad x = -0.5708, -2.6652$$

$$\sin(\theta) = \sin(3x-3) = -1 \quad \Rightarrow \quad \theta = 3x - 3 = -\pi/2, 3\pi/2$$
$$\Rightarrow \quad x = 1-\pi/6, 1+\pi/2 \quad \Rightarrow \quad x = 0.4764, 2.5708$$
$$\sin(\theta) = \sin(3x-3) = -1 \quad \Rightarrow \quad \theta = 3x - 3 = -5\pi/2$$
$$\Rightarrow \quad x = 1-5\pi/6 = -1.6180$$




Now graph $y = \left|2\sin\left(3x-3\right)+x\right| \quad \Rightarrow y \geq 0$



Graph the function $y = x\, e^{-x}$ in the domain $-\pi \le x \le \pi$

How to approach the problem:

$y = ?\quad x = 0 \quad x \to +\infty \quad x \to -\infty$

Find the critical points (turning points) $dy / dx = 0$

Use your calculator to find $y$ for a few values of $x$.

Answer:

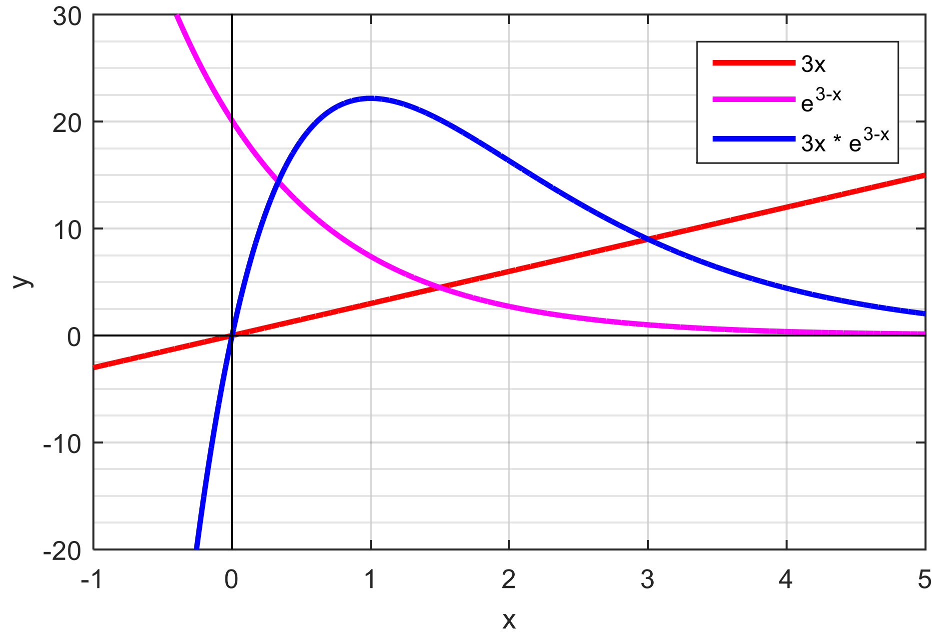
critical point $dy/dx=0 \quad (x=0 \quad y=0) \quad (x\to+\infty \quad y\to 0) \quad (x\to-\infty \quad y\to-\infty)$

$$y = x\,e^{-x} \qquad dy/dx = 3e^{3-x}(1-x) \qquad d^2y/dx^2 = (3x-6)e^{3-x}$$

$$dy/dx=0 \quad\Rightarrow\quad x=1 \quad\Rightarrow\quad y=22.1672$$

$d^2y/dx^2|_{x=1} = (-3)e^2 < 0 \quad\Rightarrow$ critical point is a maximum at $x = 1 \quad y = 22.1672$

Using calculator for $(x,y) \Rightarrow (-1,164)\,(0,0)\,(1,22.2)\,(2,16.30)\,(3,9.0)\,(4,4.4)\,(5,2.0)$



Graph the function $y = \dfrac{x^2 + 10x}{x - 2}$ How to approach the problem:

$y = ? \quad x = 0 \quad x \to +\infty \quad x \to -\infty$

Find the critical points (turning points) $dy/dx = 0$

Use your calculator to find $y$ for a few values of $x$.

Answer:

numerator – parabola $u = x^2 + 10x$

denominator- straight lines $v = x - 2$

$x = 0 \quad \Rightarrow \quad y = 0$

$y = \dfrac{x^2 + 10x}{x - 2} \quad \Rightarrow \quad y = \dfrac{x + 10/x}{1 - 2/x}$ if $x$ is very large $y \approx x$

$x \to +\infty \quad \Rightarrow \quad y \to +\infty \qquad x \to -\infty \quad \Rightarrow \quad y \to -\infty$

$x = 2 \quad \Rightarrow \quad y \to \pm\infty$ **vertical asymptote**

$\Rightarrow$ the $y$ is not a continuous function, there is a discontinuity at $x = 2$

$x = 1.90 \; y = -226 \quad x = 1.99 \; y = -2386 \quad \Rightarrow \quad x^- \to 2 \;\; y \to -\infty$

$x = 2.10 \; y = 254 \quad x = 2.01 \; y = 2414 \quad \Rightarrow \quad x^+ \to 2 \;\; y \to +\infty$

Need to find any **critical points**

$$dy/dx = 0 \quad d^2y/dx^2 < 0 \Rightarrow \max \quad d^2y/dx^2 > 0 \Rightarrow \min$$

$$y = u\,v \quad dy/dx = u\;dv/dx + v\,du/dx$$

$$u = x^2 + 10\,x \quad du/dx = 2\;x + 10$$

$$v = \left(x-2\right)^{-1} \quad dv/dx = -1/\left(x-2\right)^2$$

$$dy/dx = \frac{x^2 - 4x - 20}{\left(x-2\right)^2}$$

$$dy/dx = 0 \;\Rightarrow\; x^2 - 4x - 20 = 0 \;\Rightarrow\; x_1 = 6.89898 \quad x_2 = -2.89898$$

$$dy/dx = \frac{x^2 - 4x - 20}{\left(x-2\right)^2}$$

$$u = x^2 - 4x - 20 \quad du/dx = 2x - 4$$

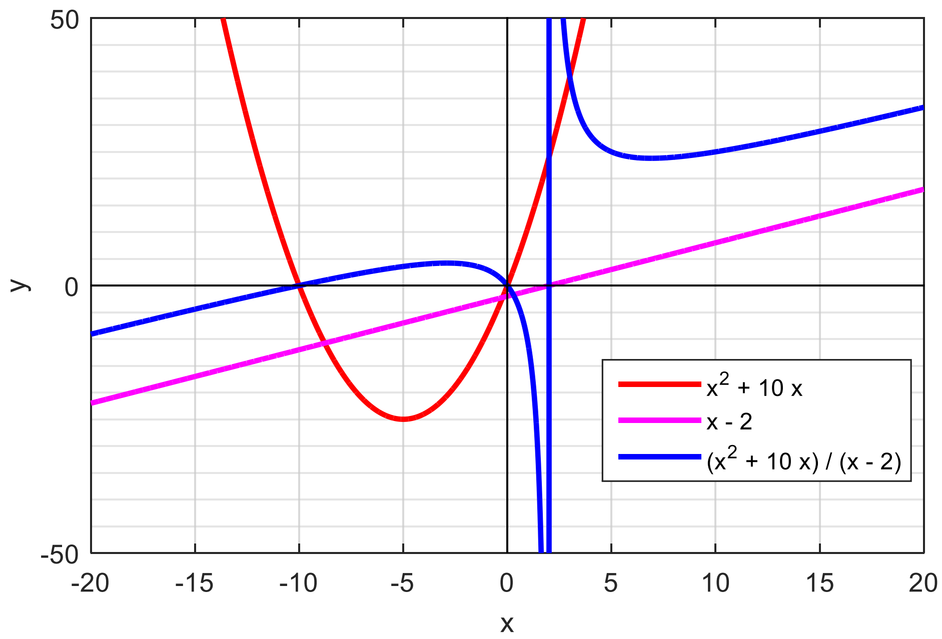
$$v = \left(x-2\right)^2 \quad dv/dx = -2/\left(x-2\right)^2$$

$$d^2y/dx^2 = u\;dv/dx + v\,du/dx$$

$$d^2y/dx^2 = \frac{48}{\left(x-2\right)^2}$$

$$x_1 = 6.89898 \quad d^2y/dx^2 = 0.4082 > 0 \quad\Rightarrow\quad \min \quad y_1 = 23.8$$

$$x_2 = -2.89898 \quad d^2y/dx^2 = -0.4082 < 0 \quad\Rightarrow\quad \max \quad y_2 = 4.2$$



50
0
-50
y
-20
-15
-10
-5
0
5
10
15
20
x
x^2 + 10 x
x - 2
(x^2 + 10 x) / (x - 2)

# INEQUALITIES

In the manipulation of inequalities, one has to take care and a little thought.

- $a > b \quad \Rightarrow \quad b < a$
- $a - b > c \quad \Rightarrow \quad a > b + c$
- $a > b \quad \Rightarrow \quad -a < -b$
- $a > 0 \quad \Rightarrow \quad 1/a > 0 \qquad a < 0 \quad \Rightarrow \quad 1/a < 0$
- $a > b > 0 \quad \Rightarrow \quad 1/b > 1/a > 0$
- $a > b \quad c \geq d \quad \Rightarrow \quad a + c > b + d$
- $a > 0 \quad b > 0 \quad \Rightarrow \quad ab > 0$
- $a > 0 \quad b < 0 \quad \Rightarrow \quad ab < 0$
- $a < 0 \quad b < 0 \quad \Rightarrow \quad ab > 0$

## Example

Show three different XY graphs, the lines or regions for the relationships:

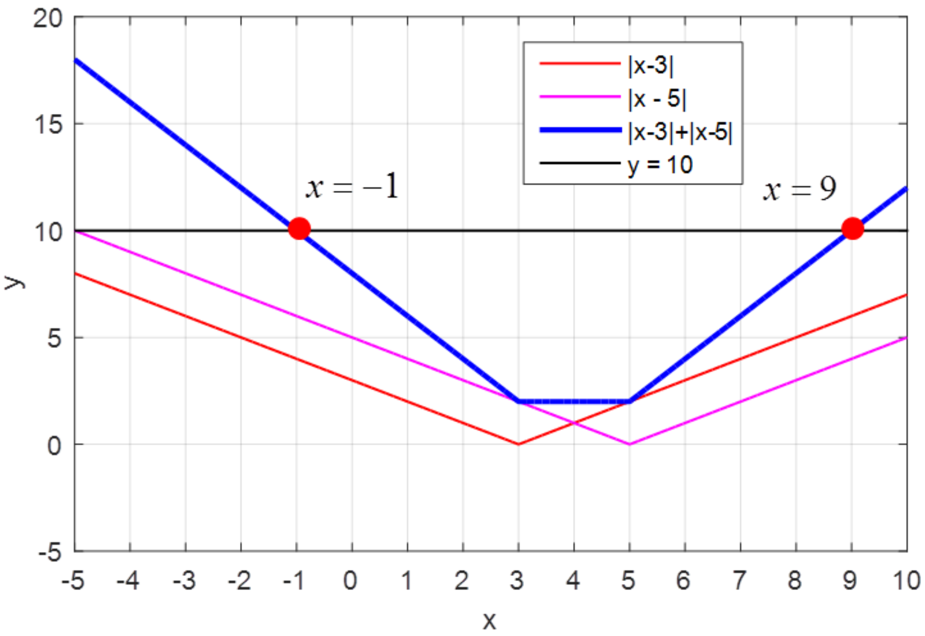
$$|x-3|+|x-5|=10 \qquad |x-3|+|x-5|\le 10 \qquad |x-3|+|x-5|>10$$

## Answers

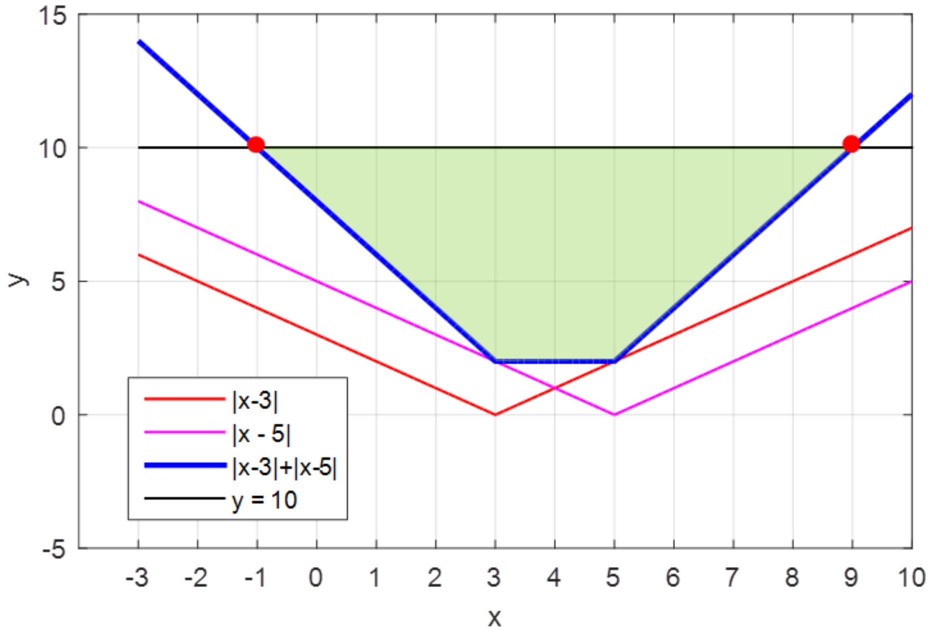
We can find the answers by graphical methods. Plot the functions

$$y_1 = |x-3| \qquad y_2 = |x-5| \qquad y_{12} = |x-3| + y = |x-5| \qquad y_3 = 10$$

The $x$ values for $|x-3| + |x-5| = 10$ are given by the points of intersection of the functions $y_{12}$ and $y_3$ $\quad \Rightarrow x = -1 \quad x = 9$



$$|x-3|+|x-5|\le 10 \quad \Rightarrow \quad |x-3|+|x-5|\le y\le 10$$



$$|x-3|+|x-5|\geq 10 \quad \Rightarrow \quad |x-3|+|x-5|\geq y\geq 10$$