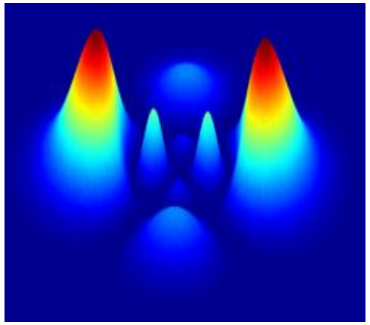


ADVANCED HIGH SCHOOL MATHEMATICS

GRAPH THEORY

SKETCHING FUNCTIONS

It is a very useful skill to be able to draw a sketch of a function $y = f(x)$ showing its essential features without actually having to draw an accurate graph.

We will consider the function

$$y = \frac{x^2 - 3x + 2}{x^2 + 3x + 2}$$

to illustrate a procedure that you should follow in curve sketching.

1. Find the value of y when x = 0

$$x = 0 \quad y = 1$$

2. Find (approximately if necessary) the values of x when $y = 0$.

$$y = 0 \quad \Rightarrow \quad x^2 - 3x + 2 = 0 \quad \Rightarrow \quad x = +1 \quad x = +2$$

3. Are they any values of x for which y becomes infinite (denominator = 0)?

$$x^2 + 3x + 2 = 0 \quad \Rightarrow \quad x = -1 \quad x = -2 \quad \Rightarrow \quad y = \pm\infty$$

We must now examine the values of y around the zeros of the numerator

$$x = -1 \Rightarrow y = \pm\infty \quad x = -1.10 \Rightarrow y = -73 \quad x = -1.01 \Rightarrow y = -611$$
$$x = -0.99 \Rightarrow y = +589 \quad x = -0.90 \Rightarrow y = +50$$

If x is a little smaller than -1, y approaches $-\infty$ but if x is a little larger than -1, y approaches $+\infty$.

$$x = -2 \Rightarrow y = \pm\infty \quad x = -2.10 \Rightarrow y = +116 \quad x = -2.01 \Rightarrow y = +1195$$
$$x = -1.99 \Rightarrow y = -1205 \quad x = -1.90 \Rightarrow y = -126$$

If x is a little smaller than -2, y approaches $+\infty$ but if x is a little larger than -2, y approaches $-\infty$.

The lines x = -1 and x = -2 are called **vertical asymptotes**.

4. Investigate what happens when $x \to \pm\infty$

$$y = \frac{1 - 3/x + 2/x^2}{1 + 3/x + 2/x^2} \quad x \to -\infty \quad \Rightarrow \quad y \to 1 \quad (\text{from above } y > 1)$$

Thus as $x \to -\infty$ the value of y decreases to 1 as the curve approaches the line y = 1. Such a line is called the **horizontal aysmptote**.

$$y = \frac{1 - 3/x + 2/x^2}{1 + 3/x + 2/x^2} \quad x \to +\infty \quad \Rightarrow \quad y \to 1 \quad (\text{from below } y < 1)$$

Thus as $x \to +\infty$ the value of y increases to 1 as the curve approaches **horizontal asymptote** y = 1.

5. Determine the **critical points** or **turning points** $dy/dx = 0$

$$dy/dx = \frac{6\left(x^2 - 2\right)}{\left(x^2 + 3x + 2\right)^2}$$

$$dy/dx = \frac{6\left(x^2 - 2\right)}{\left(x^2 + 3x + 2\right)^2} = 0 \quad \Rightarrow \quad x = \pm\sqrt{2}$$

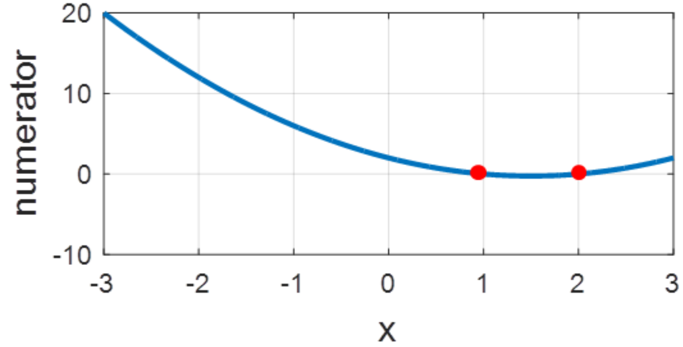
Turning points are approximately at the points $\left(-\sqrt{2}, -34\right)$ and $\left(+\sqrt{2}, -0.03\right)$

$$d^2y/dx^2 = \frac{12\left(-x^3+6x+6\right)}{\left(x^2+3x+2\right)^3}$$

$$x = -\sqrt{2} \quad d^2y/dx^2 < 0 \quad \Rightarrow \quad \text{maximum}$$
$$x = +\sqrt{2} \quad d^2y/dx^2 > 0 \quad \Rightarrow \quad \text{minimum}$$

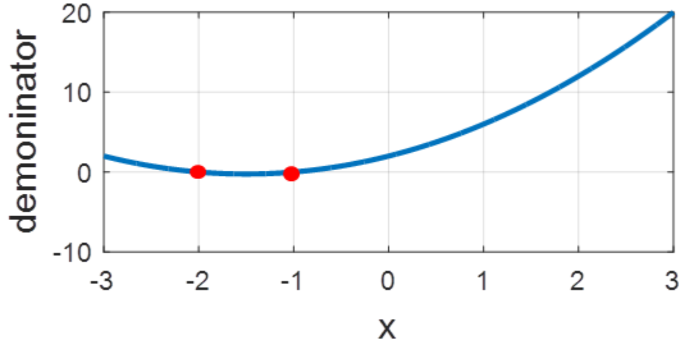
$$d^2y/dx^2 = 0 \quad \Rightarrow \quad x \approx 2.9 \quad \Rightarrow \quad \text{point of inflection}$$

6. Use your calculator to find y for a range of x values.



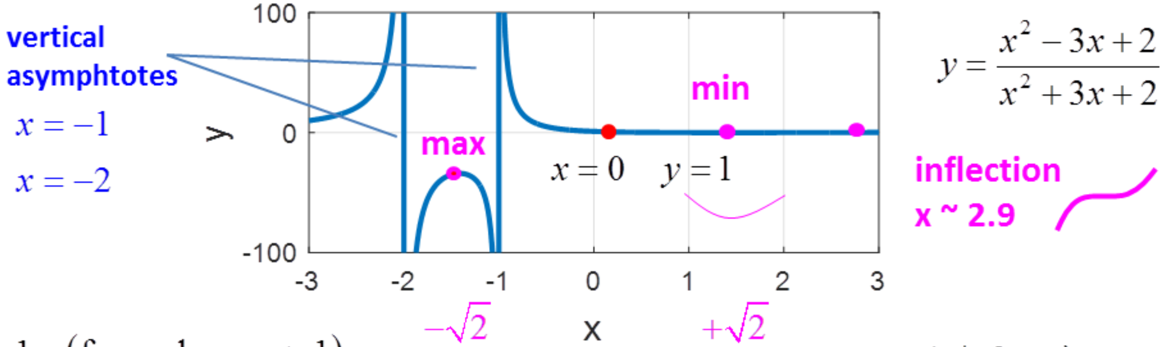
$$y_{\text{den}} = x^2 - 3x + 2$$

$$y_{\text{den}} = 0 \quad x = +1 \quad x = +2$$



$$y_{\text{num}} = x^2 + 3x + 2$$

$$y_{\text{num}} = 0 \quad x = -1 \quad x = -2$$



$$y = \frac{x^2 - 3x + 2}{x^2 + 3x + 2}$$

horizontal asymptote

$$x \to -\infty \quad \Rightarrow \quad y \to 1 \quad (\text{from above } y > 1)$$

horizontal asymptote

$$x \to +\infty \quad \Rightarrow \quad y \to 1 \quad (\text{from below } y < 1)$$