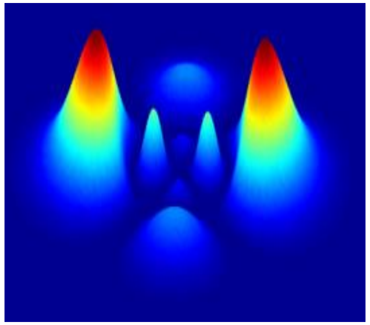


# ADVANCED HIGH SCHOOL MATHEMATICS

## GRAPH THEORY

## SLOPE OF A CURVE: MAXIMA AND MINIMA

---

Before you start this Module, review the Module on DIFFERENTIATION

Differentiation

The derivative of a function $y = f(x)$ at $x = x_1$ gives the slope (gradient) of the curve (or slope of the tangent) at the point $x_1$. Often, we are interested whether the slope of the curve is increasing or decreasing as $x$ increases or whether the curve is convex or concave towards the X-axis. A positive slope indicates that $y$ increases as $x$ increases, whereas, a negative slope indicates that $y$ decreases as $x$ increases. A zero slope indicates that $y$ does that change with increasing $x$. Thus, the sign of the first derivative $dy/dx$ is a useful indicator of the shape of the curve.

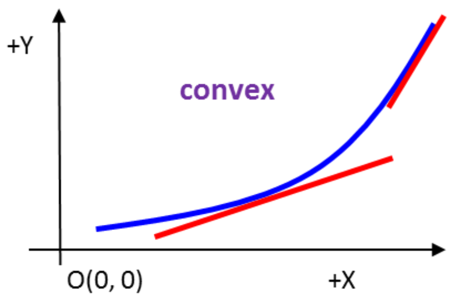
In mathematics, a **critical point** or **stationary point** or **turning point** of a differentiable function of a real or complex variable is any value in its domain where its derivative is 0 or undefined.

**critical point** or **stationary point** or **turning point** $\quad \dfrac{dy}{dx} = 0$

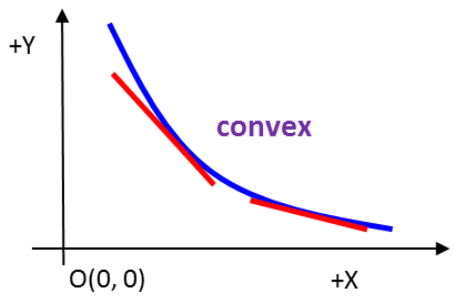
The second derivative $d^2y/dx^2$ also provides useful information. It is an indicator of the change in slope as $x$ increases and whether the curve is convex or concave towards the X-axis.

$d^2y/dx^2 > 0 \quad \Rightarrow$ slope $(dy/dx)$ of the curve increases as $x$ increases and the curve is **convex** towards the X-axis

$dy/dx > 0 \quad d^2y/dx^2 > 0$

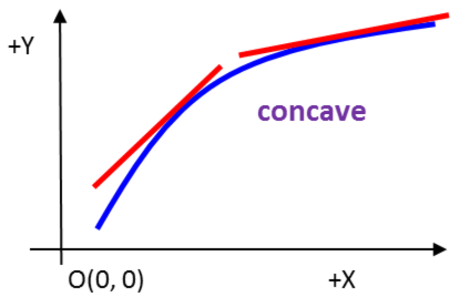



$dy/dx < 0 \quad d^2y/dx^2 > 0$

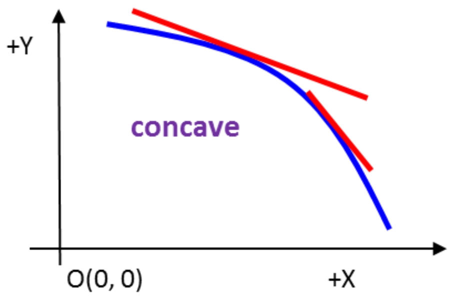



$d^2y/dx^2 < 0 \quad \Rightarrow$ slope $(dy/dx)$ of the curve decreases as $x$ increases and the curve is **concave** towards the X-axis

$dy/dx > 0 \quad d^2y/dx^2 < 0$

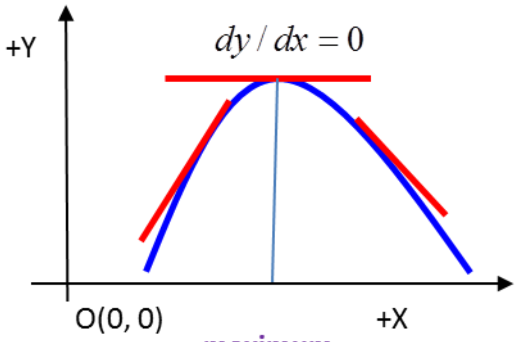



$dy/dx < 0 \quad d^2y/dx^2 < 0$



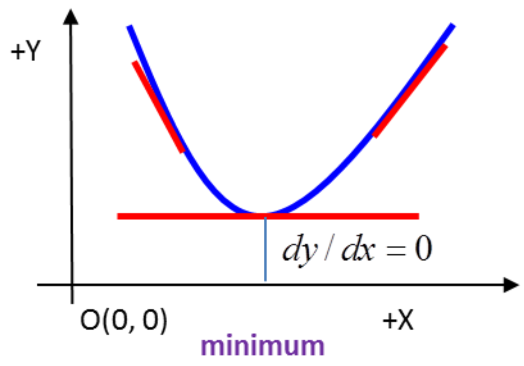
If $d^2y/dx^2 = 0$ (tangent to the curve is horizontal) then there are three possibilities about the shape of the curve. Such points are called **turning points, critical points or stationary points**.

A **maximum** occurs when the slope decreases as $x$ increases from a positive value to **zero** and then becomes negative ($dy/dx$ decreases as $x$ increases) $\Rightarrow$ $d^2y/dx^2 < 0$. At the point of the maximum $dy/dx = 0$.

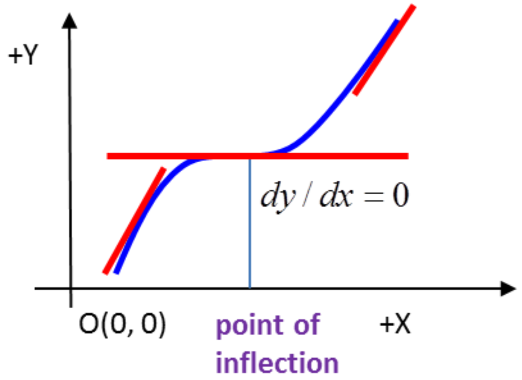


**maximum**

A **minimum** occurs when the slope increases as $x$ increases from a negative value to **zero** and then becomes positive ($dy/dx$ increases as $x$ increases) $\Rightarrow \quad d^2y/dx^2 > 0$. At the point of the minimum $dy/dx = 0$.




At a **point of inflection**, the slope decreases to **zero** and then starts to increase as $x$ increases. To the left of the point of inflection $d^2y/dx^2 < 0$ and to the right $d^2y/dx^2 > 0$ hence, at the point of inflection $d^2y/dx^2 = 0$. It is possible for $d^2y/dx^2 = 0$ without $dy/dx$ being zero. This corresponds to a point of inflection since the curve is changing from being concave upwards to being concave downwards.



$y = x^3 - 5x^2 + 2x + 8$

Roots of cubic polynomial $\alpha = -1 \quad \beta = 2 \quad \gamma = 4$

Turning points of function $y$: maximum and minimum

$dy/dx = 3x^2 - 10x + 2 = 0$

Roots of quadratic polynomial $\alpha = 0.21 \quad \beta = 3.12$

Maximum at $x = 0.21 \quad dy/dx = 0 \quad d^2y/dx^2 < 0$

Minimum at $x = 3.12 \quad dy/dx = 0 \quad d^2y/dx^2 > 0$

Turning point of parabola $dy/dx = 3x^2 - 10x + 2$

$d^2y/dx^2 = 6x - 10 = 0 \quad \Rightarrow \quad x = 1.67$

$d^3y/dx^3 = 6 > 0$

Minimum at $x = 1.67 \quad dy/dx = 0 \quad d^2y/dx^2 > 0$