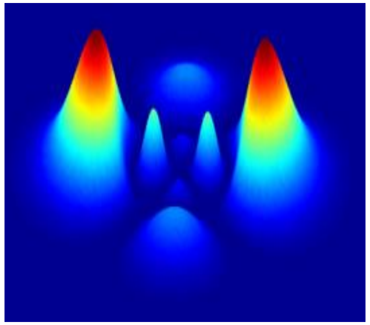


ADVANCED HIGH SCHOOL MATHEMATICS

GRAPH THEORY

GRAPHS AND LINEAR FUNCTIONS

FUNCTIONS

The concept of a function is already familiar to you. Since this concept is fundamental to mathematics, science and engineering we will briefly review it.

When we say that ***y* is a function of *x***, we mean that if we take the value x_1 then there is a corresponding value y_1. Thus, a **function** is a rule for associating a number y_1 with each number x_1.

y* is a function of *x $\quad x_1 \to y_1 \quad \Rightarrow \quad y = f(x) \quad y = y(x)$

In mathematics the symbols x and y are used too often. Consider the function describing the Stefan-Boltzmann equation which relates the surface temperature of on object to the net power radiated / absorbed from that surface.

$$P = \varepsilon \sigma A\left(T^4 - T_o^{\,4}\right)$$

In a functional relationship you must always distinguish between the symbols representing the **variables** and the symbols representing **constants**. For the Stefan-Boltzmann equation

P	power (variable)
T	surface temperature of the surface (variable)
T_0	temperature of environment surrounding object (constant)
ε	characteristic of the surface (constant)
σ	Stefan-Boltzmann constant (constant)
A	surface area of object (constant)

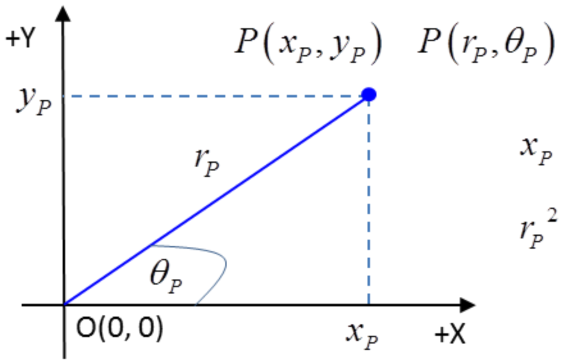
To gain insight to a functional relationship, the variables are often plotted against each other to create a **graph**. The graph of P (y variable) against T^4 (x variable) is a straight line.

The variable x is often called the independent variable because we can select a value of x and then associate with it a value of y, the dependent variable. In the sciences and engineering, it is good practice **never** use the terms independent variable and dependent variable, always just consider the functional relationship between the variables.

GRAPHICAL REPRESENTATION OF FUNCTIONS

A convenient representation of a function $y = f(x)$ is a **graph** which uses a **right-angled Cartesian coordinate system** labelled the **abscissa** (horizontal X-axis) and the **ordinate** (vertical Y-axis). The axes intersect at the point called the origin O which has the Cartesian coordinates (0, 0).

The **Cartesian coordinates** of a point P are usually written as $\left(x_P, y_P\right)$.The point P can also be located on a graph using **polar coordinates** $\left(r_P, \theta_P\right)$ where r_P is the distance OP and θ_P is the angle the line OP makes with the X-axis. The use of polar coordinates is important in plotting complex numbers (Topic 2) on Argand Diagrams (XY graph: X-axis: real part of the complex number and Y-axis: complex part of the real number).



$$x_P = r_P \cos\left(\theta_P\right) \quad y_P = r_P \sin\left(\theta_P\right)$$

$$r_P{}^2 = x_P{}^2 + y_P{}^2 \quad \tan\left(\theta_P\right) = \frac{y_P}{x_P}$$

The simplest type of function is the **linear function**

(1) $$a\,x + b\,y + c = 0$$

where x and y are the variables and a, b and c are the constants. In a linear function, the variables are only raised to the **first** power. Equation (1) is not the most useful way of expressing a linear function. The most useful expression for a linear relationship is given by equation (2)

(2) $y = m\,x + b$ variables x, y constants m, b

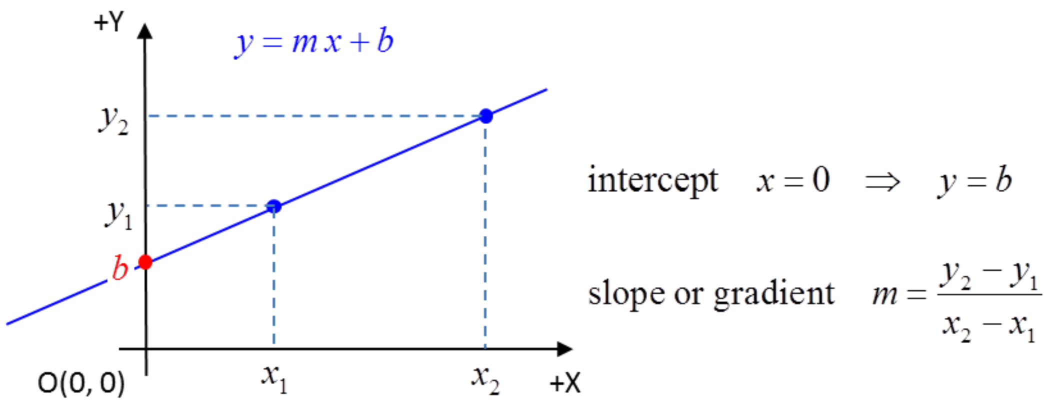
The graph of a linear function is a **straight line**. The **intercept** b on the Y-axis is the y value at x = 0. If we take two points on the straight line with coordinates (x_1, y_1) and (x_2, y_2) then the **slope** m or **gradient** of the straight line is defined by

(3) $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ slope (gradient)

$$slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{m\,x_2 + b - (m\,x_1 + b)}{x_2 - x_1} = m$$

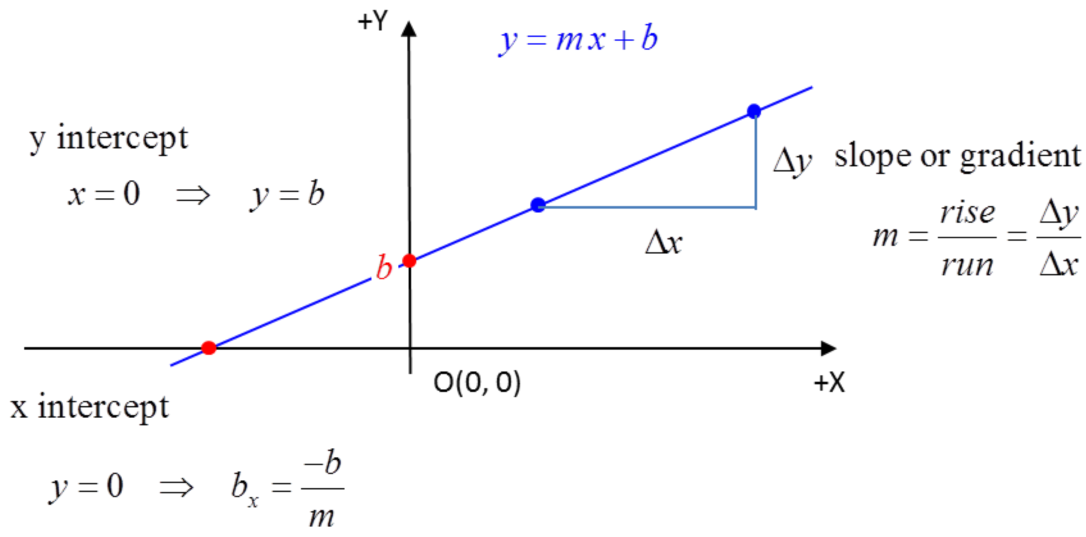
If we differentiate equation (2) with respect to x, then the derivative dy/dx is equal to the gradient or slope of the straight line

(4) $dy/dx = m$ slope (gradient)



The gradient can also be expressed as

(5) $$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x}$$



Linear graphs are very important in the analysis of data because they are characterised by two parameters m and b and it is easy to see if a set of data points lie on a straight line, whereas, it is difficult to decide if a set of points corresponds to a particular curve. In data analysis, wherever possible we try to convert a function to a linear function in drawing a graph to establish relationships between variables. For example, in the Stefan-Boltzmann equation, P plotted against T is a curved line, however by plotting P against T^4 we get a straight line.

Linear Relationship and straight line graph $y = m\,x + b$

- X-axis $y = 0$
- Y-axis $x = 0$
- Straight line parallel to the X-axis $y = b$ $m = 0$
- Straight line parallel to the Y-axis $x = b_x$ $m = \infty$
- Two parallel lines $m_1 = m_2$
- Two perpendicular lines (lines at right angles to each other)

  $m_1\,m_2 = -1 \quad m_1 = \dfrac{-1}{m_2} \quad m_2 = \dfrac{-1}{m_1}$
- If two lines $y = m_1\,x + b_1$ and $y = m_2\,x + b_2$ intersect at the point $P\left(x_P, y_P\right)$ then

  $y_P = m_1\,x_P + b_1 = m_2\,x_P + b_2$

Example

For $-15 < x < 15$

- Plot the function $y = -2x + 10$
- Plot the function $y = 3x - 5$
- At the point x = - 4, plot the straight line which is perpendicular to the line $y = -2x + 10$
- Calculate the Cartesian coordinates of the three intersection points P, Q and R for the three straight lines.

To show that two lines are perpendicular in your plot the X and Y axes must have the same scale.

Solution

To plot a straight line graph, you only need to select the Cartesian coordinates for two points:

$$y = -2x + 10 \quad x_1 = 0 \quad y_1 = 10 \quad x_2 = 5 \quad y_2 = 0$$

$$y = 3x - 5 \quad x_1 = 0 \quad y_1 = -5 \quad x_2 = 5 \quad y_2 = 10$$

For perpendicular line:

$$y = -2x + 10 \quad m_1 = -2 \quad b_1 = 10$$

$$\perp \text{ line}$$

$$y = m_2 x + b_2 \quad m_2 = -1/m_1 = 1/2 \quad b_2 = y - x/2$$

$$\text{intersection point} \quad x = -4 \quad y = -2x + 10 = (-2)(-4) + 10 = 18$$

$$b_2 = y - x/2 = 18 - (-4)/2 = 20$$

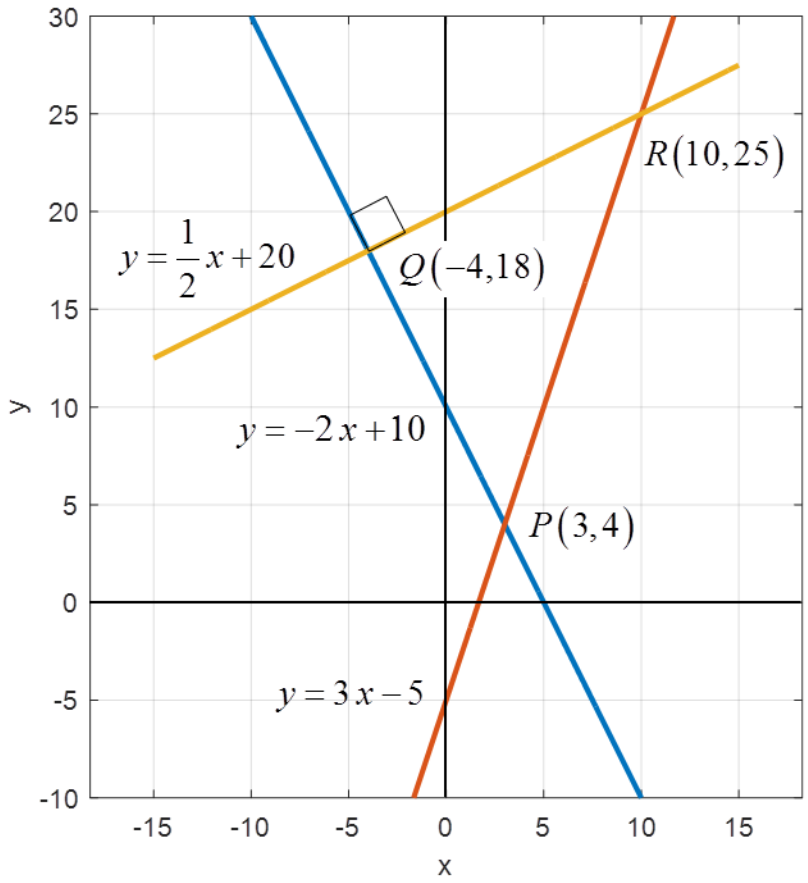
$$y = x/2 + 20 \quad x = -4 \quad y = 18$$

Intersection points

$$y = -2x + 10 = 3x - 5 \qquad x = 3 \quad y = 4$$

$$y = -2x + 10 = x/2 + 20 \qquad x = -4 \quad y = 18$$

$$y = 3x - 5 = x/2 + 20 \qquad x = 10 \quad y = 25$$



30
25
20
15
10
5
0
-5
-10
y
-15
-10
-5
0
5
10
15
x
$R(10,25)$
$y = \frac{1}{2}x + 20$
$Q(-4,18)$
$y = -2x + 10$
$P(3,4)$
$y = 3x - 5$

Example

Find the equation of the linear function through the points (-3, 6) and (6, -3)

Solution

Equation of a linear function: $y = mx + b$

Substitute in the coordinates for the two points: $(A) \quad 6 = -3m + b \quad (B) \quad -3 = 6m + b$

Solve for m and b: Eq (A) – Eq(B) $9 = -9m \quad \Rightarrow \quad m = -1 \quad b = 3$

The linear relationship is $y = -x + 3$

Alternatively:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-6}{6-(-3)} = -1$$

$$y = -x + b \quad b = y_1 + x_1 = -3 + 6 = 3$$

$$y = -x + 3$$

MORE ON FUNCTIONS

A polynomial is a function of the form

$$y = f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n = \sum_{i=0}^{n} a_i x^i$$

The **degree of the polynomial** is ***n*** (n integer $n = 0, 1, 2, \ldots$). Such a function is defined for all values of x and x is finite.

A linear function (n = 1) is a polynomial of degree 1.

A polynomial of degree 2 (n = 2) is called a **quadratic function**

$$y = a_0 + a_1 x + a_2 x^2$$

The quadratic function is mostly expressed as

$$y = a x^2 + b x + c$$

The graph of a quadratic function is a **parabola**. If there are real values of x for which $y = 0$, the parabola will intersect the X-axis at

real roots $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad b^2 - 4ac \geq 0$

Polynomial functions are called **single-valued** functions because there is only one value of $\boldsymbol{y}$ for each value of $\boldsymbol{x}$. The function $y^2 = x$ is a **multi-valued** function since there are two values of y for each value of x: $+\sqrt{x_1}$ and $-\sqrt{x_1}$

Functions can depend upon a number of variables. For example, the pressure p of a gas in a container depends upon the volume V of the container and the temperature T of the gas.

$$p = \frac{nRT}{V} \qquad \text{variabels}\left(p, T, V\right) \qquad \text{constants}\left(n, R\right)$$

This is an example of an **explicit function**, since the equation can be rearranged to make the variables V or T the subject of the equation

$$p = \frac{nRT}{V} \qquad V = \frac{nRT}{p} \quad T = \frac{pV}{nR} \qquad \textbf{explicit function}$$

This is not the case for the equation below in regard to the variable V. This is an example of an **implicit function**

$$\left(p + \frac{n^2 a}{V^2}\right)\left(V - nb\right) = nRT \qquad \textbf{implicit function}$$

A useful classification of functions is into even and odd functions.

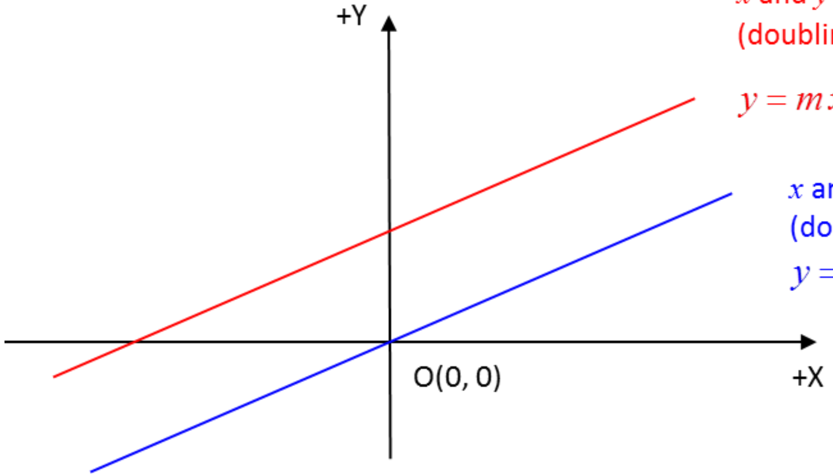
An **even function** of x is one that remains unchanged when the sign of x is reversed

$$f(-x) = f(x) \qquad \text{even function}$$

whereas an **odd function** changes sign

$$f(-x) = -f(x) \qquad \text{odd function}$$

Many students misinterpret the terms **proportional** (directly proportional) and **inverse proportional**. They conclude that if y increases as x increases then x and y are proportional to each other and if y decreases as increases then x and y are inversely proportional. These conclusions are **wrong**.

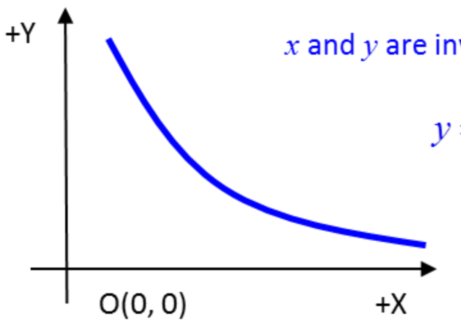


x and y are **not** directly proportional to each other (doubling the value of x does **not** double the value of y)

$$y = mx + b \quad b \neq 0$$

x and y are directly proportional to each other (doubling the value of x doubles the value of y)

$$y = mx \quad \Rightarrow \quad y \propto x$$



x and y are inversely proportional to each other only if

$$y = \frac{k}{x} \quad k = \text{constant}$$

doubling the value of x then the value of y is halved

"All students" studying mathematics know that $y = m\,x + b$ is the equation of a straight line and $y = x^2$ is the equation of a parabola. But what about the equations

$v = u + a\,t$ and $s = u\,t + \frac{1}{2}a\,t^2$ **? ? ?**

Sadly, the majority of students doing physics don't recognize that $v = u + a\,t$ is also a straight line and $s = u\,t + \frac{1}{2}a\,t^2$ is a parabola. These two equations describe an object moving with a constant acceleration.

The variables are t (time), v (velocity at time t) and s (displacement at time t, $t = 0$ $s = 0$) while the constants are u (initial velocity, $t = 0$, $v = u$) and a (constant acceleration).

velocity $v = \dfrac{ds}{dt}$ $\Rightarrow$ velocity = slope of s/t graph

acceleration $a = \dfrac{dv}{dt}$ $\Rightarrow$ acceleration = slope of v/t graph

$a = \text{constant} \;\Rightarrow\; \dfrac{da}{dt} = 0 \;\Rightarrow$ slope of a/t graph = 0

$$\text{straight line} \quad y = m\,x + b \quad \Leftrightarrow v = u + a\,t$$

$$y \Leftrightarrow v \quad x \Leftrightarrow t \quad b \Leftrightarrow u \quad m \Leftrightarrow a$$

$$\text{parabola} \quad y = a\,x^2 + b\,x + c \quad \Leftrightarrow s = u\,t + \tfrac{1}{2}a\,t^2$$

$$y \Leftrightarrow s \quad x \Leftrightarrow t \quad a \Leftrightarrow \tfrac{1}{2}a \quad b \Leftrightarrow u \quad c = 0$$

To improve your understanding in interpreting graphs you should do the online Activity

Simulation – Workshop – Uniform Acceleration