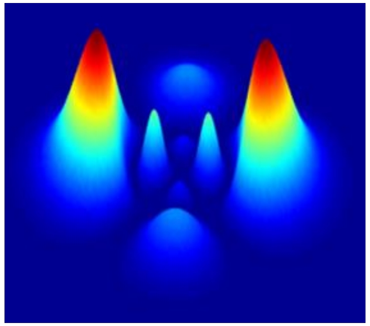


ADVANCED HIGH SCHOOL MATHEMATICS

DIFFERENTIATION

Differentiation is concerned with the rates of change of physical quantities. It is a fundamental topic in mathematics, physics, chemistry, engineering etc.

Consider a continuous and single value function $y = f(x)$. The rate of change of y with respect to x at the point x_1 is called the **derivative** and equals the slope of the tangent to the curve $y = f(x)$ at the point x_1. The process of finding the derivative of a function is called **differentiation**.

Take two points P(x_1, y_1) and Q(x_2, y_2) on the curve $y = f(x)$. We require the slope of the tangent at the point P(x_1, y_1). The slope of the straight line (chord) joining the points P and Q is

$$\text{slope PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Let $x_2 = x_1 + \Delta x$, $\Delta x = x_2 - x_1$ and $f(x_2) = f(x_1 + \Delta x)$. The point Q approaches the point P as $\Delta x \to 0$ and the slope of the chord approaches the slope of the tangent at the point x_1.

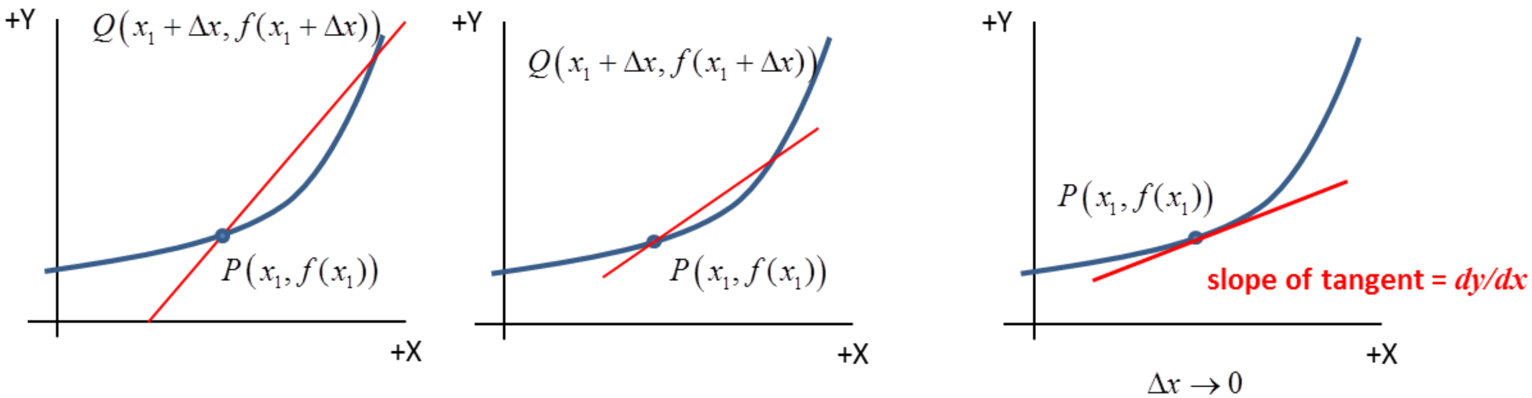
Intuitively, we can say that the slope of the tangent at P will be given by the limit of the slope of the chord as $\Delta x \to 0$

$$\text{slope at P} = \underset{x \to 0}{\text{limit}} \left(\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \right)$$

Given a curve $y = f(x)$ and a point P on the curve, the slope of the curve at P is the limit of the slope of lines between P and Q on the curve as Q approaches P. The slope of a curve $y = f(x)$ is the rate at which y is changing as x changes or it is the rate of change of y with respect to x. This **slope** is known as the **derivative of the function y with respect to x**. It is given by the special symbols

$$\frac{dy}{dx} \qquad \frac{d\,f(x)}{dx} \qquad f\,'(x) \qquad \dot{y}$$

$$\frac{dy}{dx} = \underset{x\to 0}{\text{limit}}\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)$$



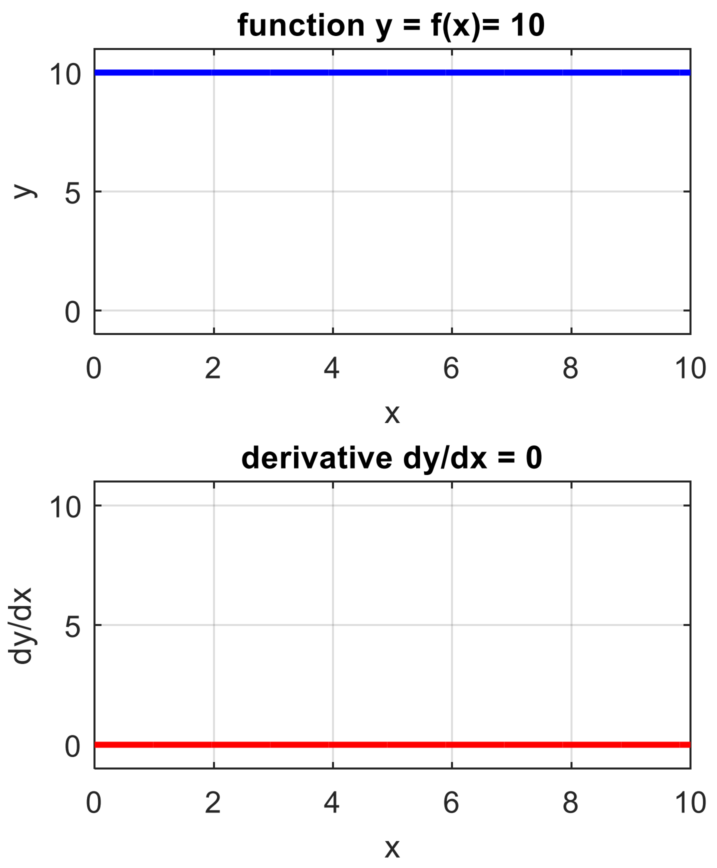
$$\text{slope chord PQ} = \frac{f(x_1+\Delta x) - f(x_1)}{x_2 - x_1}$$

RULES FOR DIFFERENTIATION

The derivative of a constant is zero

$$y = \text{constant}$$

$$\frac{dy}{dx} = 0$$



The derivative of powers of x

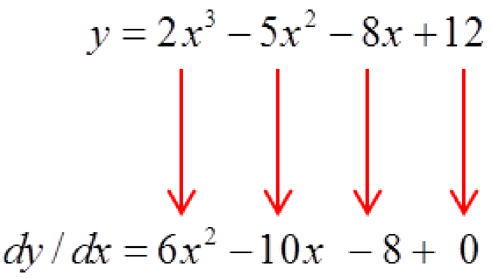
$$y = A x^n$$

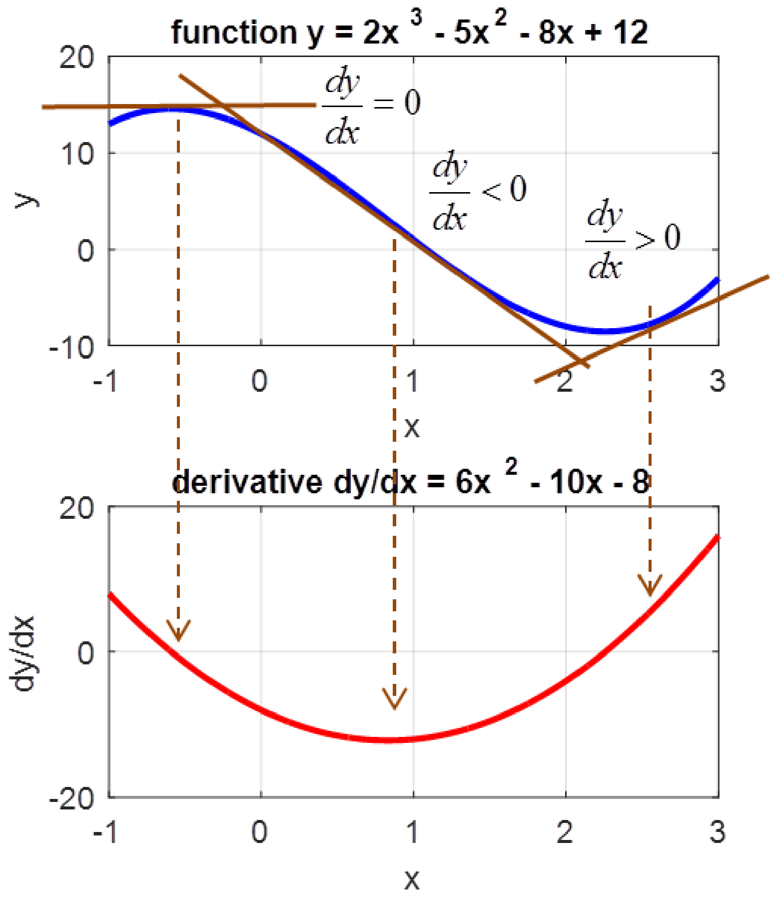
$$\frac{dy}{dx} = n\, A x^{n-1}$$

Example

$$y = 2x^3 - 5x^2 - 8x + 12$$

$$dy / dx = 6x^2 - 10x - 8$$





Proof

$$f(x) = 2x^3 - 5x^2 - 8x + 12$$

$$f(x+\Delta x) = 2\left(x+\Delta x\right)^3 - 5\left(x+\Delta x\right)^2 - 8\left(x+\Delta x\right)x + 12$$

$$f(x+\Delta x) = 2x^3 - 5x^2 - 8x + 12 + \Delta x\left(6x^2 - 10x - 8\right) + \Delta x^2\left(6x + 2\Delta x - 5\right)$$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \left(6x^2 - 10x - 8\right) + \Delta x\left(6x + 2\Delta x - 5\right)$$

$$\underset{x\to 0}{\text{limit}}\left\{\frac{f(x+\Delta x) - f(x)}{\Delta x}\right\} = \frac{dy}{dx} = 6x^2 - 10x - 8$$

The derivative of a product

$$u = f_1(x) \qquad v = f_2(x)$$

$$y = f_1(x)\ f_2(x)$$

$$y = u\ v$$

$$\frac{dy}{dx} = f_1(x)\,\frac{d}{dx}\,f_2(x) + f_2(x)\,\frac{d}{dx}\,f_1(x)$$

$$\frac{dy}{dx} = u\,\frac{dv}{dx} + v\,\frac{du}{dx}$$

This rule can be extended to the product of several functions

$$u = f_1(x) \qquad v = f_2(x) \qquad w = f_3(x)$$

$$y = u\,v\,w$$

$$\frac{dy}{dx} = u\,v\,\frac{dw}{dx} + u\,w\,\frac{dv}{dx} + v\,w\,\frac{du}{dx}$$

Example

$$y = \left(3x^6 + 4x^{-1/2}\right)\left(3x^2 + 6x^{1/2} + 8\right)$$

$$u = \left(3x^6 + 4x^{-1/2}\right) \quad du/dx = 18x^5 - 2x^{-3/2}$$

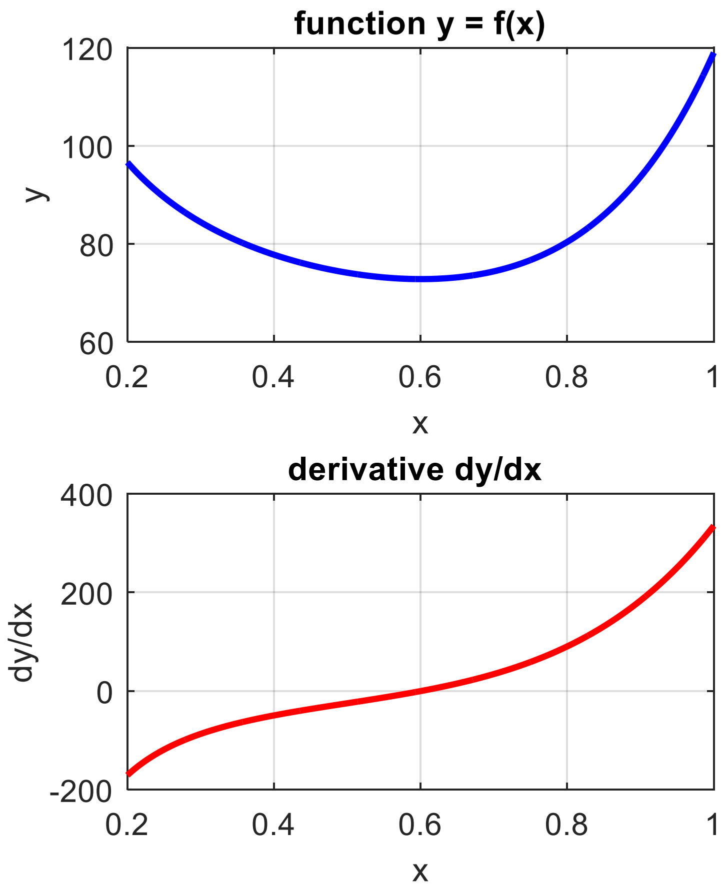
$$v = \left(3x^2 + 6x^{1/2} + 8\right) \quad dv/dx = 6x + 3x^{-1/2}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$dy/dx = \left(3x^6 + 4x^{-1/2}\right)\left(6x + 3x^{-1/2}\right) + \left(3x^2 + 6x^{1/2} + 8\right)\left(18x^5 - 2x^{-3/2}\right)$$

$$dy/dx = \left(18x^7 + 24x^{1/2} + 9x^{11/2} + 12x^{-1}\right) + \left(54x^7 + 108x^{11/2} + 144x^5 - 6x^{1/2} - 12x^{-1} - 16x^{-3/2}\right)$$

$$dy/dx = 72x^7 + 117x^{11/2} + 144x^5 + 18x^{1/2} - 16x^{-3/2}$$



The chain rule – differentiation of a function of a function

If $y = f(u)$ and $u = g(x)$ then the derivative of y with respect to x is given by

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example

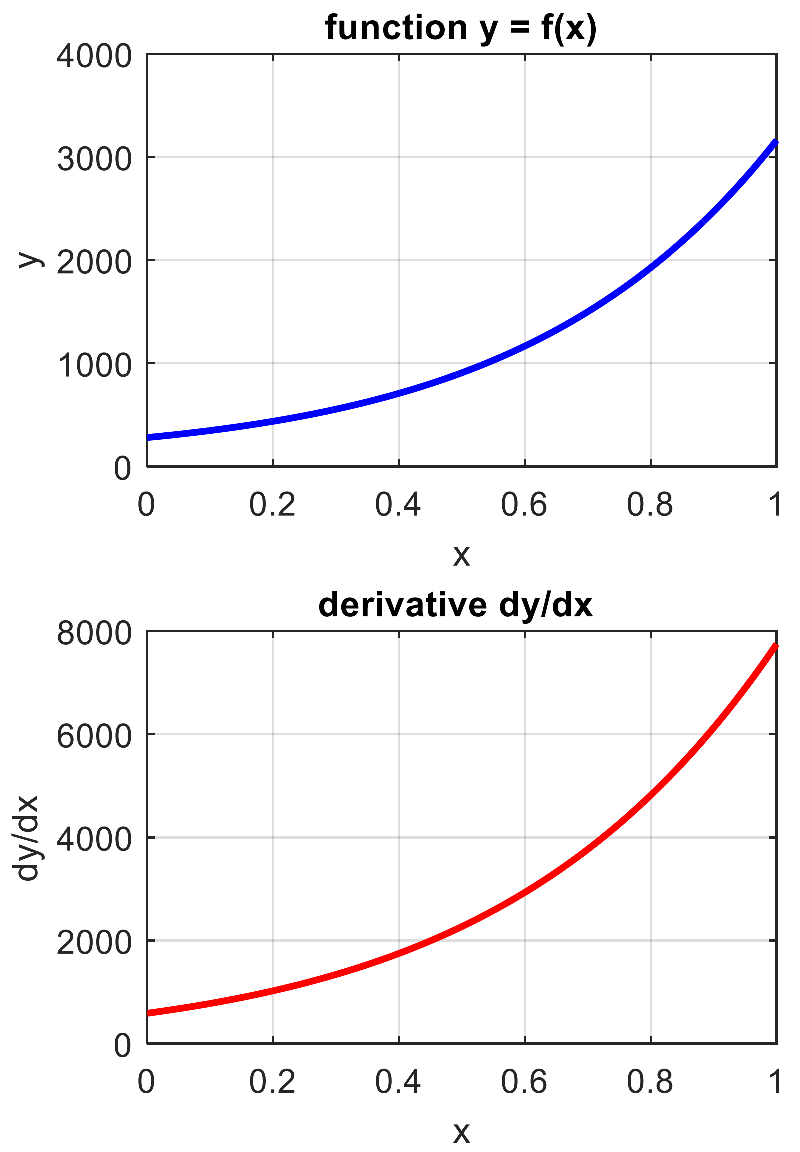
$$y = \left(2x^2 + 3x + 5\right)^{7/2}$$

$$u = 2x^2 + 3x + 5 \qquad du/dx = 4x + 3$$

$$y = u^{7/2} \qquad dy/du = \left(7/2\right)u^{5/2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(\frac{7}{2}\right)u^{5/2}\left(4x+3\right)$$

$$\frac{dy}{dx} = \left(\frac{7}{2}\right)\left(2x^2 + 3x + 5\right)^{5/2}\left(4x+3\right)$$



The derivative of a quotient

The derivative of the quotient $y = \frac{f(x)}{g(x)} = \frac{u}{w} \quad u = f(x) \quad x = g(x)$ provided that $g(x) \neq 0$ is given by

$$\frac{dy}{dx} = \frac{w\left(du/dx\right) - u\left(dw/dx\right)}{w^2}$$

I think it often better to use only the **product rule** and not the quotient rule

$$y = \frac{u}{w}$$

$$w = \frac{1}{v}$$

$$y = u\,v$$

$$\frac{dy}{dx} = u\,\frac{dv}{dx} + v\,\frac{du}{dx}$$

Proof

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{w}\right) = \frac{d}{dx}\left(u\,w^{-1}\right)$$

$$\frac{dy}{dx} = u\left(\frac{-1}{w^2}\right)\frac{dw}{dx} + \left(\frac{1}{w}\right)\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{w\left(du/dx\right) - u\left(dw/dx\right)}{w^2}$$

Example

$$y = \sqrt{\frac{2x+1}{2x-1}}$$

$$y = \sqrt{\frac{2x+1}{2x-1}} = \left(2x+1\right)^{1/2}\left(2x+1\right)^{-1/2}$$

$$u = \left(2x+1\right)^{1/2} \qquad du/dx = \left(2x+1\right)^{-1/2}$$

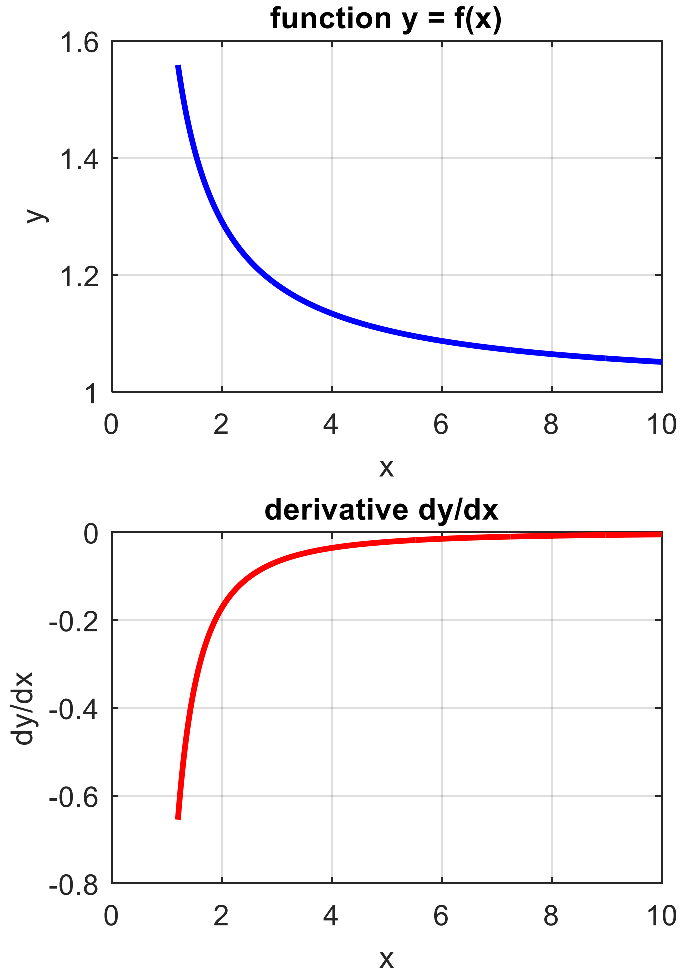
$$v = \left(2x-1\right)^{-1/2} \qquad dv/dx = -\left(2x-1\right)^{-3/2}$$

$$y = u\,v$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = \left(2x+1\right)^{1/2}\left(-\left(2x-1\right)^{-3/2}\right) + \left(2x-1\right)^{-1/2}\left(2x+1\right)^{-1/2}$$

$$\frac{dy}{dx} = \frac{\left(2x-1\right)^{3/2} - \left(2x+1\right)\left(2x-1\right)^{1/2}}{\left(2x-1\right)^{2}\left(2x+1\right)^{1/2}}$$



Differentiation of trigonometric functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{1}{\tan x}\right) = \operatorname{cosec}^2 x$$

Examples

$$y = \sin^2 x$$

$$u = \sin x \quad y = u^2 \quad dy/du = 2u \quad du/dx = \cos x$$

$$dy/dx = (dy/du)(du/dx) = (2u)(\cos x)$$

$$dy/dx = 2\sin x \cos x$$

$$y = \frac{\sin x}{x}$$

$$u = \sin x \quad du/dx = \cos x \quad v = x^{-1} \quad dv/dx = -x^{-2}$$

$$y = u\,v \quad dy/dx = u\,dv/dx + v\,du/dx$$

$$dy/dx = (\sin x)(-x^{-2}) + = (x^{-1})(\cos x)$$

$$dy/dx = \frac{x\cos x - \sin x}{x^2}$$

Differentiation of inverse trigonometric functions

If $y = f(x)$ then the inverse function is $x = g(y)$ then

$$\frac{dg(y)}{dy} = \frac{1}{df(x)/dx} \qquad \frac{dx}{dy} = \frac{1}{dy/dx}$$

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2 - x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1}\left(\frac{x}{a}\right)\right) = \frac{-1}{\sqrt{a^2 - x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{a^2 + x^2}$$

Examples

$$y = \sin^{-1}\left(\frac{x}{a}\right)$$

$$x/a = \sin y \qquad x = a\sin y$$

$$dx/dy = a\cos y$$

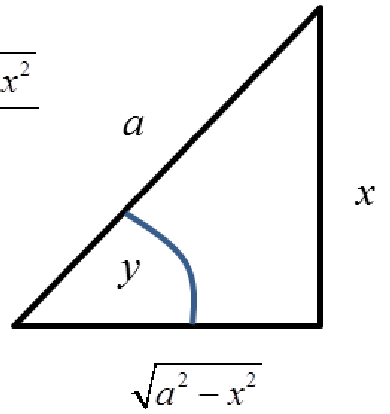
$$dy/dx = 1/dx/dy = \frac{1}{a\cos y}$$

$$\cos y = \frac{\sqrt{a^2 - x^2}}{a}$$

$$dy/dx = \frac{1}{\sqrt{a^2 - x^2}}$$

$$\sin y = \frac{x}{a}$$

$$\cos y = \frac{\sqrt{a^2 - x^2}}{a}$$



$$y = \cos^{-1}\left(\frac{x}{a}\right)$$

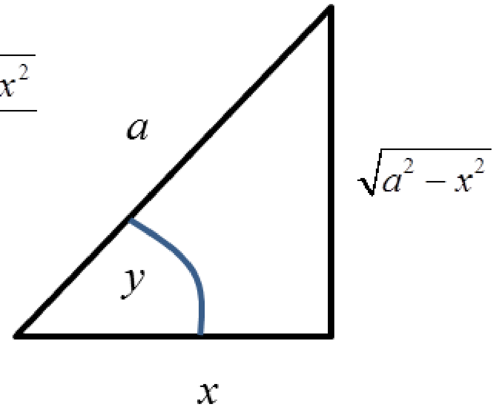
$$\cos y = \frac{x}{a}$$

$$x/a = \cos y \qquad x = a\cos y$$

$$\sin y = \frac{\sqrt{a^2 - x^2}}{a}$$

$$dx/dy = -a\sin y$$

$$dy/dx = 1/dx/dy = \frac{-1}{a\sin y}$$



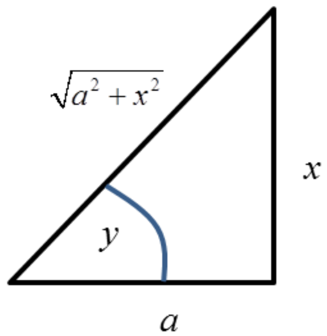
$$\sin y = \frac{\sqrt{a^2 - x^2}}{a}$$

$$dy/dx = \frac{-1}{\sqrt{a^2 - x^2}}$$

$$\tan y = \frac{x}{a}$$

$$\cos y = \frac{a}{\sqrt{a^2 + x^2}}$$

$$\cos^2 y = \frac{a^2}{a^2 + x^2}$$



$$y = \tan^{-1}\left(\frac{x}{a}\right)$$

$$x = a \tan y = \frac{a \sin y}{\cos y}$$

$$u = a \sin y \quad du / dx = a \cos y \quad v = (\cos y)^{-1} \quad dv / dx = \sin y \,(\cos y)^{-2}$$

$$x = u\, v \quad dx / dy = u\, dv / dy + v\, du / dy$$

$$dx / dy = (a \sin y)\left(\sin y \,(\cos y)^{-2}\right) + (\cos y)^{-1} a \,(\cos y)$$

$$dx / dy = a \,\frac{\sin^2 y + \cos^2 y}{\cos^2 y}$$

$$dx / dy = \frac{a}{\cos^2 y}$$

$$dy / dx = \frac{\cos^2 y}{a}$$

$$dy / dx = \frac{a}{a^2 + x^2}$$

Differentiation of inverse exponential and logarithmic functions

$$y = a\, e^{b\,x}$$

$$dy/dx = a\, b\, e^{b\,x} = b\; y$$

$$y = a\,\log_e\left(b\,x\right) \equiv a\,\ln\left(b\,x\right)$$

$$dy/dx = \frac{a}{x}$$

$$y = a^{x}$$

$$dy/dx = a^{x}\log_e\left(a\right)$$

Examples and proofs

$$y = a \log_e(b\,x)$$

$$e^{y/a} = b\,x$$

$$x = (1/b)\,e^{y/a}$$

$$dx/dy = (1/b)(1/a)\,e^{y/a}$$

$$dy/dx = \frac{a\,b}{e^{y/a}} = \frac{a\,b}{b\,x}$$

$$dy/dx = \frac{a}{x}$$

$$y = a^x$$

$$\log_e(y) = \log_e(a^x) = x \log_e(a)$$

$$x = \frac{\log_e(y)}{\log_e(a)}$$

$$dx/dy = \frac{1}{y \log_e(a)}$$

$$dy/dx = y \log_e(a)$$

$$dy/dx = a^x \log_e(a)$$

$$y = \log_e\left(x^2 + 3x + 2\right)$$

$$u = x^2 + 3x + 2 \quad du/dx = 2x + 3$$

$$y = \log_e(u) \quad dy/du = 1/u$$

$$dy/dx = (dy/du)(du/dx)$$

$$dy/dx = (1/u)(2x + 3)$$

$$dy/dx = \frac{2x + 3}{x^2 + 3x + 2}$$

$$y = \log_e\left(\frac{x}{2 + 3x}\right)$$

$$y = \log_e(x) - \log_e(2 + 3x)$$

$$dy/dx = \frac{1}{x} - \frac{3}{2 + 3x}$$

$$dy/dx = \frac{2}{x(2 + 3x)}$$

Higher derivatives

We can differentiate a function many times

Function	$y = f(x)$
1st derivative	$dy/dx = f'(x) = \dot{y}$
2nd derivative	$d^2y/dx^2 = f''(x) = \ddot{y}$

Example

$$y = 2x^5 + 3x^4 + 4x^3 + 5x^2 + 6x + 9$$

$$dy/dx = 10x^4 + 12x^3 + 12x^2 + 10x + 6$$

$$d^2y/dx^2 = 40x^3 + 36x^2 + 24x + 10$$

$$d^3y/dx^3 = 120x^2 + 72x + 24$$

$$d^4y/dx^4 = 240x + 72$$

$$d^5y/dx^5 = 240$$

$$d^6y/dx^6 = 0$$

APPLICATIONS OF DIFFERENTIATION

The derivative of the function $y = f(x)$ is the rate of change of y with respect to x. The derivative is a very useful quantity in analysing systems that change with time.

In radioactive decay, the number of nuclei N remaining time t is

$$N = N_0 \, e^{-\lambda t}$$

where N_0 is the number of nuclei at time $t = 0$ and λ is a constant called the decay constant.

The rate of decay is proportional to the number of remaining nuclei

$$\frac{dN}{dt} = -\lambda \, N_0 \, e^{-\lambda t} = -\lambda \, N$$

The minus sign indicates the number of nuclei remaining decreases with time.

The displacement s of a moving particle is a function of time t.

displacement	$s = f(t)$
velocity	$v = ds/dt$
acceleration	$a = d^2s/dt^2 = dv/dt$

Example

A particle's displacement s is a function of time t is given by the equation

$$s = 4t^3 - 3t^2 - 6t + 1$$

At what time is the acceleration of the particle zero?

$$v = ds/dt = 12t^2 - 6t - 6$$

$$a = dv/dt = 24t - 6$$

$$a = 0 \Rightarrow 24t - 6 = 0 \Rightarrow t = \frac{6}{24} = \frac{1}{4}$$