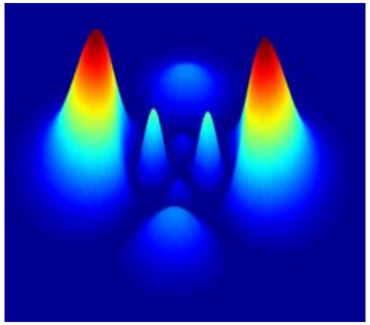


ADVANCED HIGH SCHOOL MATHEMATICS

CONICS

PARABOLA

A **parabola** is defined as the locus of a point that moves so that its distance from a fixed point is equal to its distance from a fixed line.

If the fixed point called the **focus** is $(0, a)$ and the fixed line called the **directrix** is $y = -a$ then the equation of the locus is

$$y = \frac{x^2}{4a} \qquad x^2 = 4ay$$

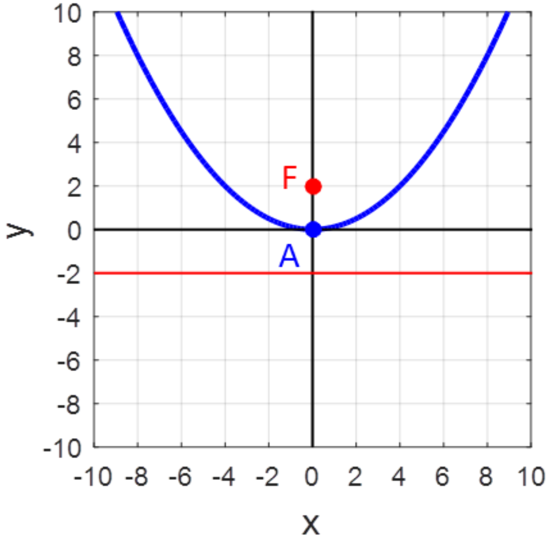
The vertex of the **parabola** is A$(0,\ 0)$. The parameter a is known as the **focal length**.

For a parabola with **vertex** A(x_0, y_0) its equation is

$$(y - y_0) = \frac{(x - x_0)^2}{4a}$$

and the **directrix** is $y_0 - a$ and the **focal point** is F$\left(x_0, y_0 + a\right)$.

$a = 2 \quad y = \dfrac{x^2}{4a} \quad \Rightarrow \quad y = \dfrac{x^2}{8}$



focus F(0, a)   F(0, 2)

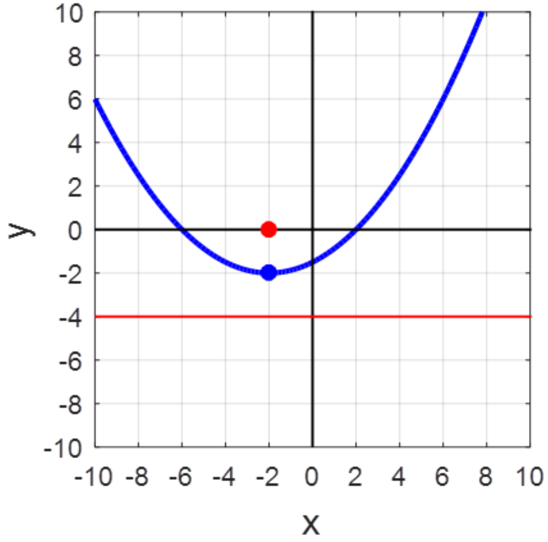
directrix y = - a y = =2

vertex A(0, 0)

$a = 2 \quad \left(y - y_0\right) = \dfrac{\left(x - x_0\right)^2}{4a} \quad \Rightarrow \quad y = y_0 + \dfrac{\left(x - x_0\right)^2}{8}$

$x_0 = -2$

$y_0 = -2$



focus F(x_0, y_0+ a)   F(0, 0)

directrix $y = y_0 - a$ y = -4

vertex A(x_0, y_0)   A(-2 -2)

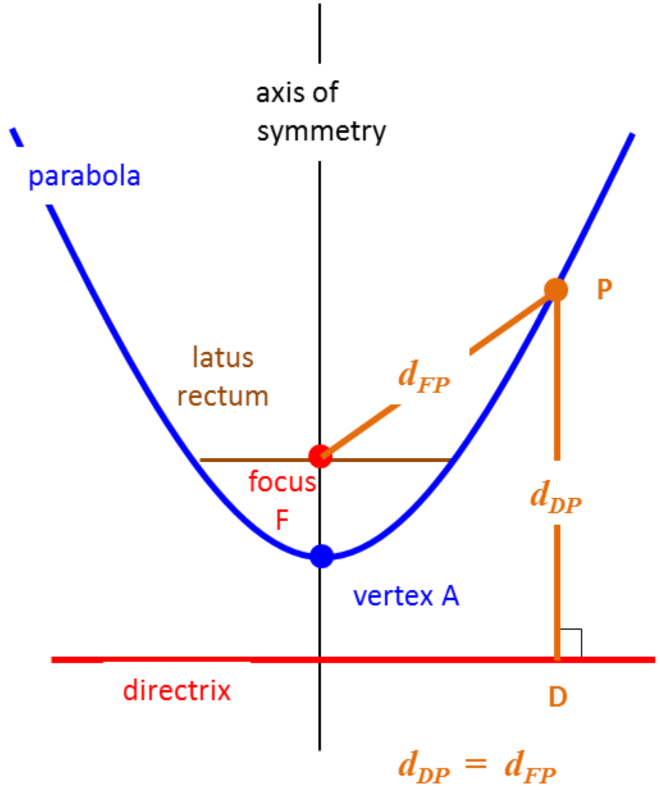
According to ancient Greeks, to form a parabola, you would start with a line (**directrix**) and a point (**focus**). The parabola is the curve formed from all the points P$\left(x_P, y_P\right)$ that are equidistant from the directrix and the focus. The **axis of symmetry** of the parabola is the line perpendicular to the directrix and passing through the focus - the line that splits the parabola up the middle. The **vertex** is the point on the axis of symmetry that is exactly midway between the focus and the directrix where the parabola changes direction.

Let the directrix be the line $y = -a$ and let the focus F be the point (0, a). If P(x_P, y_P) is a point on the parabola then, by definition of a parabola, it is the same distance from the directrix as the focus

distance of point P to the directrix $d_{DP} = |y_P + a|$

distance of point P from focus F

$$d_{FP} = \sqrt{x_P^{\,2} + (y_P - a)^2}$$



$$d_{DP} = d_{FP}$$

$$|y_P + a| = \sqrt{x_P^{\,2} + (y_P - a)^2}$$

$$y_P^{\,2} + 2\,y_P\,a + a^2 = x_P^{\,2} + y_P^{\,2} - 2\,y_P\,a + a^2$$

$$y_P = \left(\frac{1}{4a}\right) x_P^{\,2}$$

Hence the equation of a parabola is

$$y = \frac{x^2}{4a} \qquad x^2 = 4\,a\,y$$

eccentricity $\quad e = \dfrac{d_{FP}}{d_{DP}} = 1$

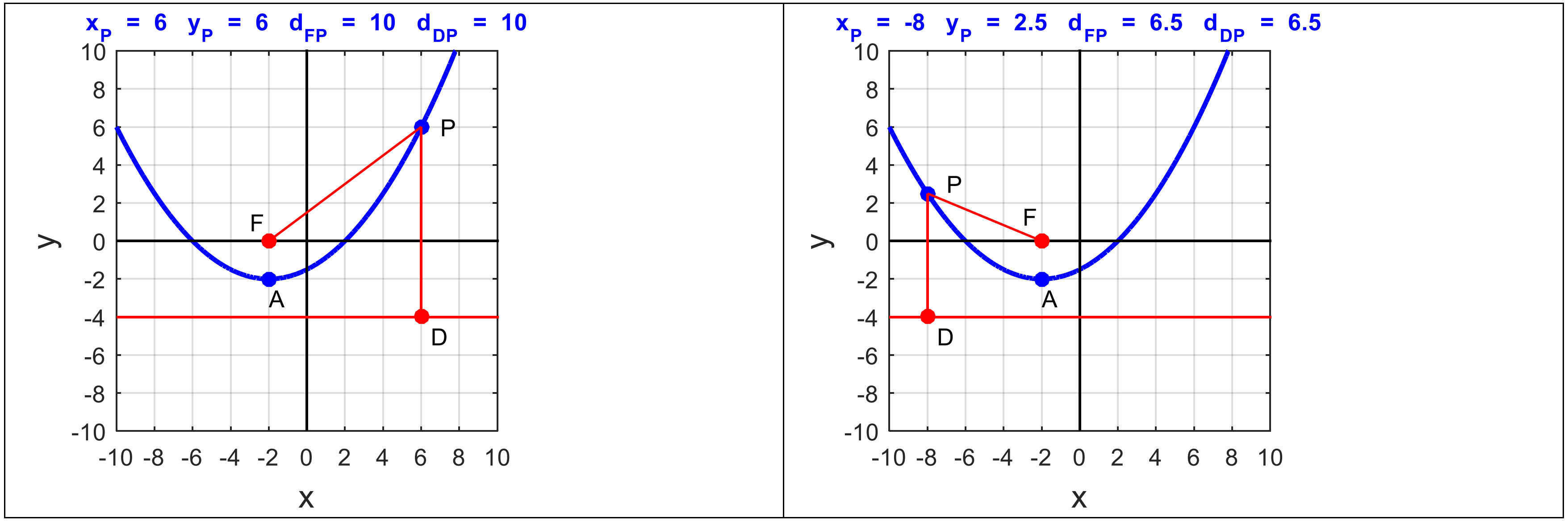
Example

$$a = 2 \quad x_0 = -2 \quad y_0 = -2$$

$$\left(y - y_0\right) = \frac{\left(x - x_0\right)^2}{4a} \quad \Rightarrow \quad y = -2 + \frac{\left(x + 2\right)^2}{8}$$

$$P\left(x_P, y_P\right) \quad F\left(x_0, y_0 + a\right) \quad D(x_P, y_0 - a)$$

$$d_{FP} = d_{DP}$$



The most general form for the equation of a parabola is

$y = a\,x^2 + b\,x + c \quad a \neq 0$ *note: a is the same quantity as used in the above equations*

Let the vertex of the parabola be A(x_0, y_0) and p be the distance from the vertex to the focus and the vertex to the directrix, then

$$(y - y_0) = \left(\frac{1}{4\,p}\right)(x - x_0)^2$$ *note: use* p *not* a

$$y = \left(\frac{1}{4\,p}\right)\left(x^2 - 2\,x_0\,x + x_0^{\,2}\right) + y_0$$

but $y = a\,x^2 + b\,x + c$

hence $a = \dfrac{1}{4\,p} \quad b = -\dfrac{x_0}{4\,p} \quad c = y_0 + \dfrac{x_0^{\,2}}{2\,p}$

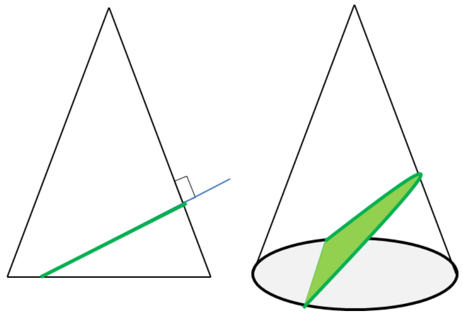
$$p = \frac{1}{4a} \quad x_0 = -\frac{b}{2\,a} \quad y_0 = \frac{4\,a\,c - b^2}{4a}$$

Parametric form of the equation for the parabola

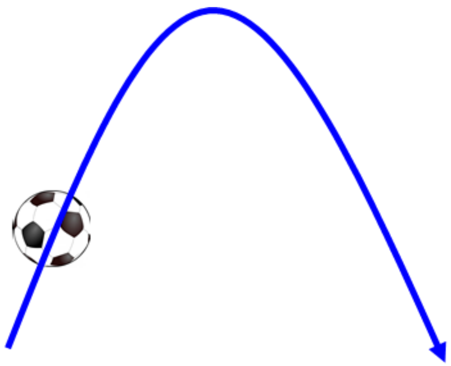
$$(y - y_0) = \left(\frac{1}{4p}\right)(x - x_0)^2$$

$$x(t) = x_0 + 2pt \quad y(t) = y_0 + pt^2$$

The parabola is a member of the family of conic sections. A parabola is obtained as the intersection of a cone with a plane parallel to a plane which is tangential to the cone's surface.

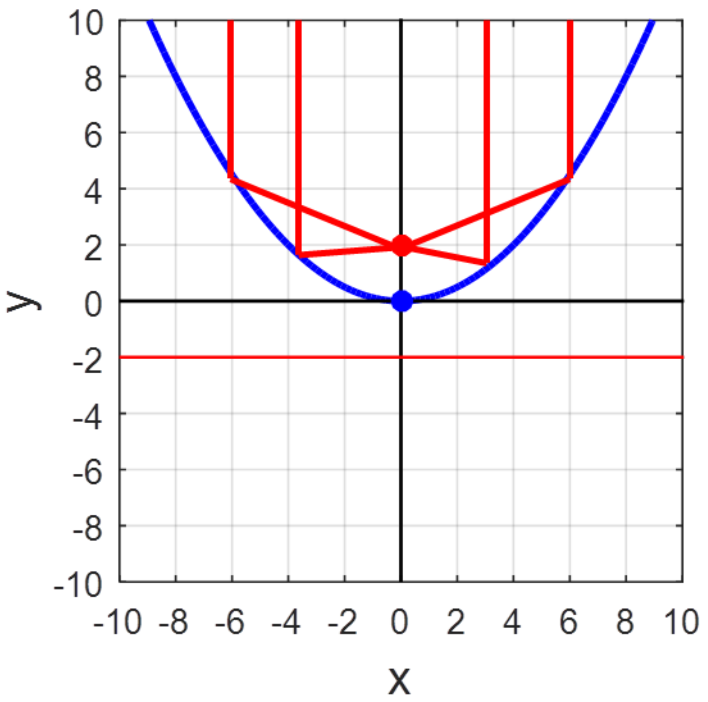


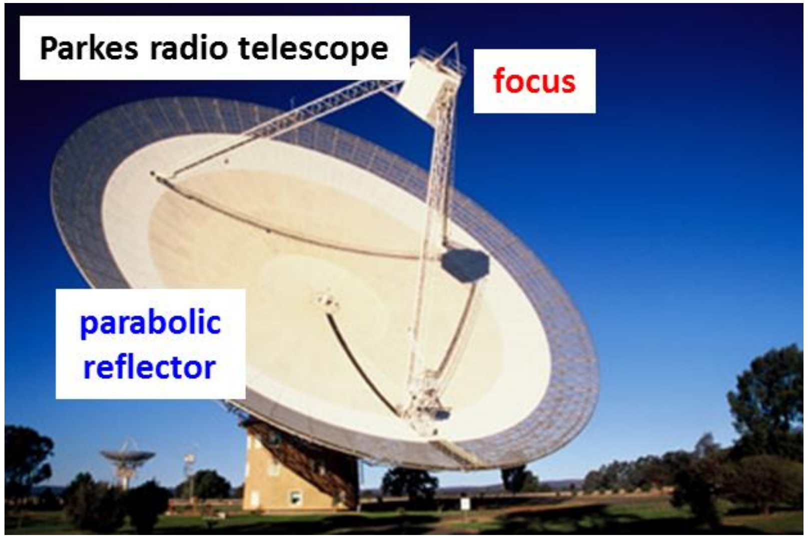
Ignoring the effect of air resistance, when you kick a soccer ball or shoot an arrow, fire a missile or throw a stone it arcs up into the air and comes down again, the projectile following the path of a parabola!



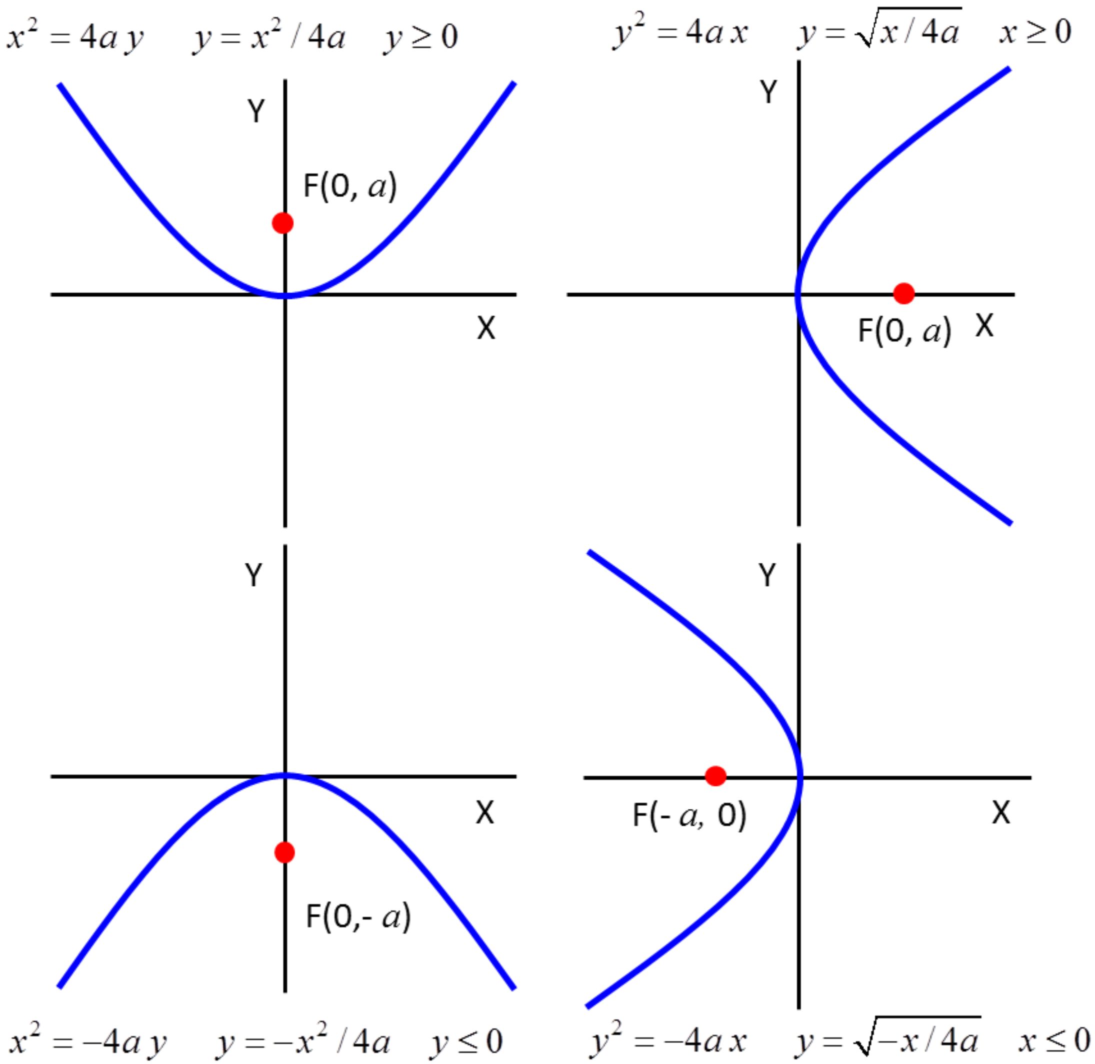
Parabolic Reflectors

A parabola has an amazing property: any ray parallel to the axis of symmetry gets reflected off the surface straight to the focus. Parabolic reflectors can be used for satellite dishes, radar dishes, concentrating the Sun's rays to make a hot spot, the reflector on spotlights and torches, etc





Parabolas

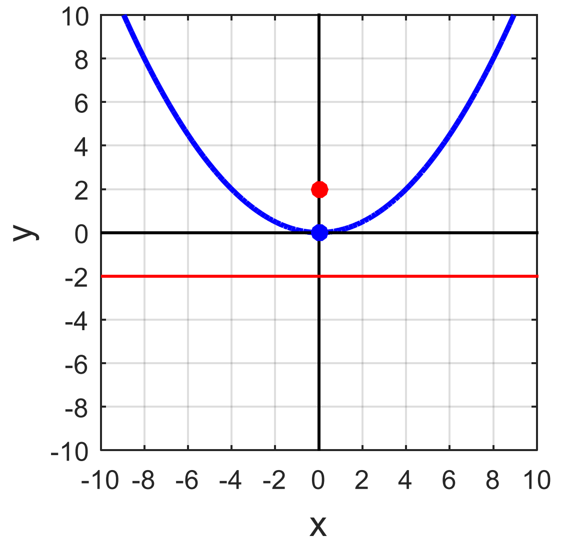

$x^2 = 4a\,y \quad y = x^2/4a \quad y \ge 0$
Y
F(0, a)
X
$y^2 = 4a\,x \quad y = \sqrt{x/4a} \quad x \ge 0$
Y
F(0, a)
X
Y
X
F(0,- a)
$x^2 = -4a\,y \quad y = -x^2/4a \quad y \le 0$
Y
F(- a, 0)
X
$y^2 = -4a\,x \quad y = \sqrt{-x/4a} \quad x \le 0$

Example Parabolic dish

A parabolic dish has a focus is 2.00 m. What is the equation of the parabola determining the shape of the dish? What are the y values of the dish for x = 0, 1, 2, 3, 4, 5, 8 m.

Solution

The equation of the parabola is $y = \left(\dfrac{1}{4a}\right)x^2$



The focal length of the parabola is a = 2.00 m

Hence, the required shape of the disk is given by $y = \dfrac{x^2}{8}$

[x = 0 y = 0] [x = 1.000 y = 0.125] [x = 2.000 y = 0.500]

[x = 3.000 y = 1.125] [x = 4.000 y = 2.000]

[x = 5.000 y = 3.125] [x = 8.000 y = 8.00]