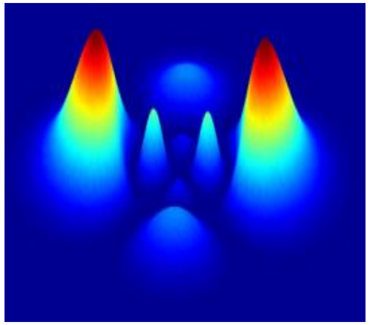


ADVANCED HIGH SCHOOL MATHEMATICS

CONICS

RECTANGULAR HYPERBOLA

A hyperbola for which the **asymptotes** are **perpendicular is** a **rectangular hyperbola** and is also called an equilateral hyperbola or right hyperbola. This occurs when the semimajor a and semiminor b axes are equal, $a = b$.

equation $\quad x^2 - y^2 = a^2$ **rectangular hyperbola opening to the left and right**

eccentricity $\quad c^2 = a^2 + b^2 \quad a = b \quad c = a\sqrt{2} \quad e = \dfrac{c}{a} = \sqrt{2}$

directrix $\quad x = \pm\dfrac{a^2}{c} = \pm\dfrac{a}{\sqrt{2}}$

asymptotes $\quad y = \pm\, x$

Example: Verify the information shown in the figure below

$a = 5$ $b = 5$ $c = 7.07$

$P(x, y) = (7.5, 5.59)$

$A_1(x, y) = (-5, 0)$ $A_2(x, y) = (5, 0)$

$F_1(x, y) = (-7.07, 0)$ $F_2(x, y) = (7.07, 0)$

$D = (3.54, 5.59)$

eccentricity $e = 1.41$

directrices 1: $x = -3.54$ directrices 2: $x = 3.54$

slope tangent $M_1 = 1.34$ slope normal $M_2 = -0.745$

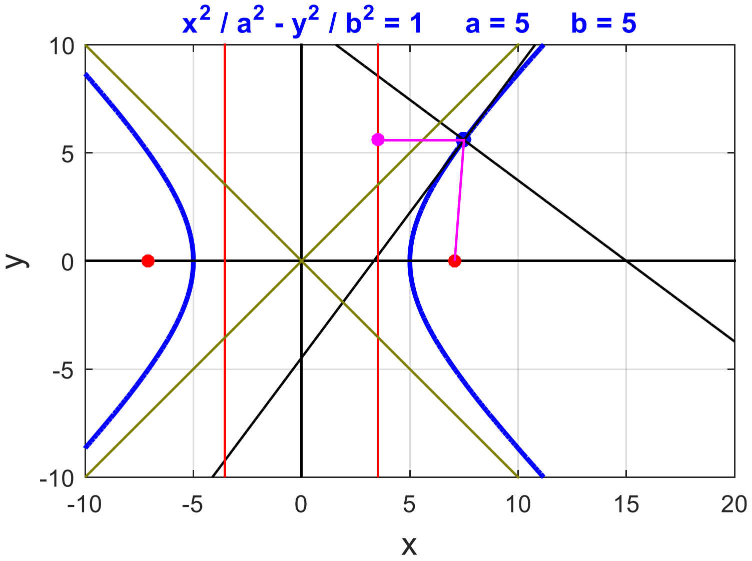
intercept tangent $B_1 = -4.47$ intercept normal $B_2 = 11.2$

T tangent cross X-axis: $x_T = 3.33$ N normal cross X-axis: $x_N = 15$

distances: $PF_1 = 15.6$ $PF_2 = 5.61$ $|PF_1 - PF_2| = 10$

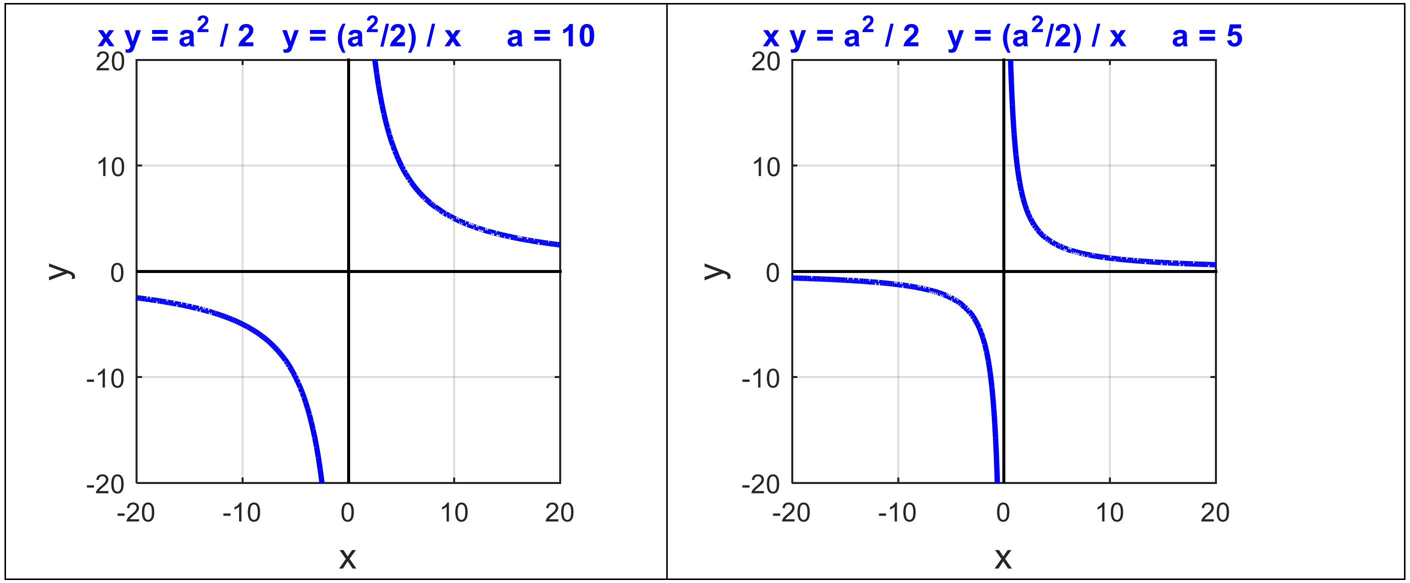
distances: $PF_2 = 5.61$ $PD = 3.96$ $PF_2 / PD = 1.41$

$x_P^2 / a^2 - y_P^2 / b^2 = 1$ asymptotes $y = 1\ x$ asymptotes $y = -1\ x$



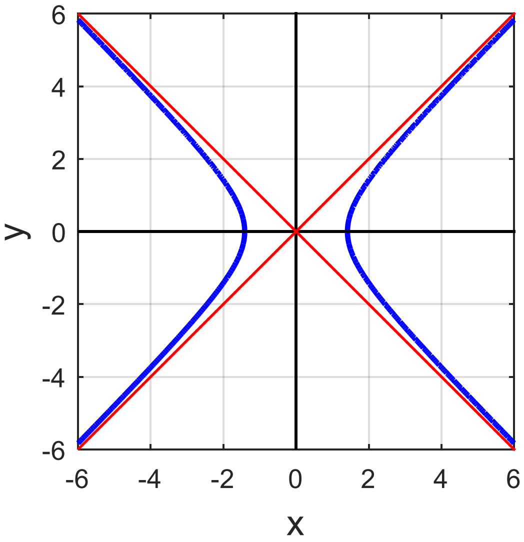
Rectangular hyperbola opening in the first and third quadrants has the Cartesian equation

$$x\,y = \frac{a^2}{2} \qquad y = \frac{a^2/2}{x}$$



$x^2 - y^2 = a^2$

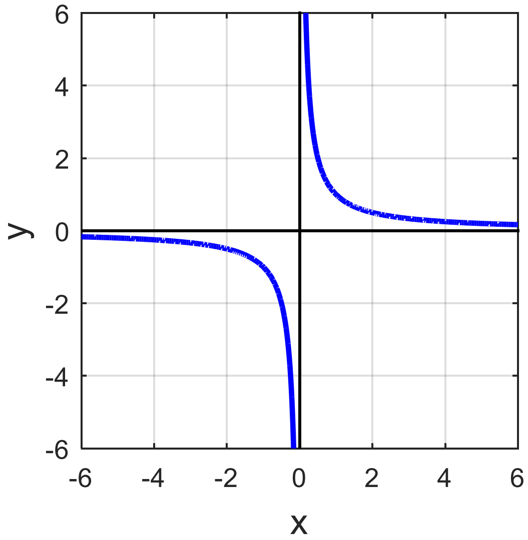
$a = \sqrt{2} \quad \Rightarrow \quad x^2 - y^2 = 2 \quad y = \pm\left(x^2 - 2\right)$



Rotate graph 45^{o} anticlockwise to give plot on right

$x\,y = \frac{a^2}{2} \quad y = \frac{a^2/2}{x}$

$a = \sqrt{2} \quad \Rightarrow \quad x\,y = 1 \quad y = 1/x$



Rotate graph 45^{o} iclockwise to give plot on right

Example: Verify the information shown in the figure below ($x^2 - y^2 = 2 \qquad a = b = \sqrt{2}$)

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a = 1.41   b = 1.41   c = 2

P(x, y) = (7.5, 7.365)

A_1(x, y) = (-1.41, 0) A_2(x, y) = (1.41, 0)

F_1(x, y) = (-2, 0) F_2(x, y) = (2, 0)

D = (1, 7.365)

eccentricity e = 1.41

directrices 1: x = -1   directrices 2: x = 1

slope tangent M_1 = 1.02   slope normal M_2 = -0.982

intercept tangent B_1 = -0.272   intercept normal B_2 = 14.7

T tangent cross X-axis: x_T = 0.267   N normal cross X-axis: x_N = 15

distances: PF_1 = 12 PF_2 = 9.19 $|PF_1 - PF_2|$ = 2.83

distances: PF_2 = 9.19   PD = 6.5 PF_2 / PD = 1.41

$x_P^2 / a^2 - y_P^2 / b^2 = 1$ asymptotes y = 1 x   asymptotes y = -1 x

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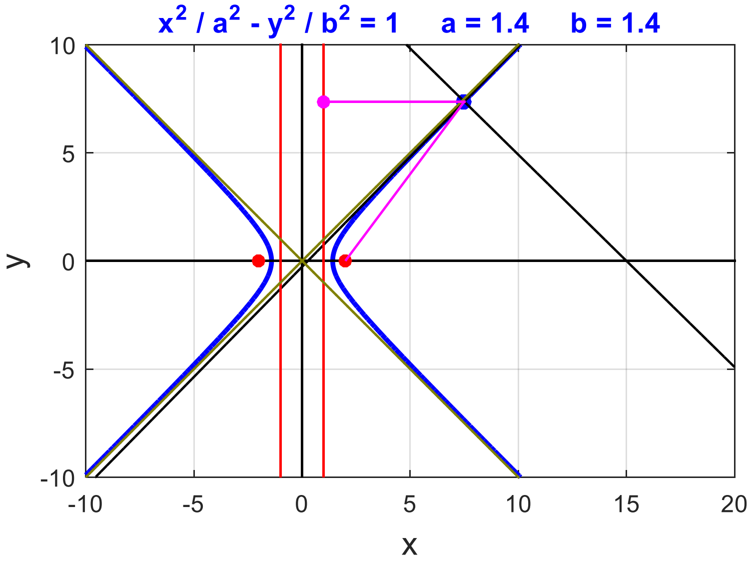
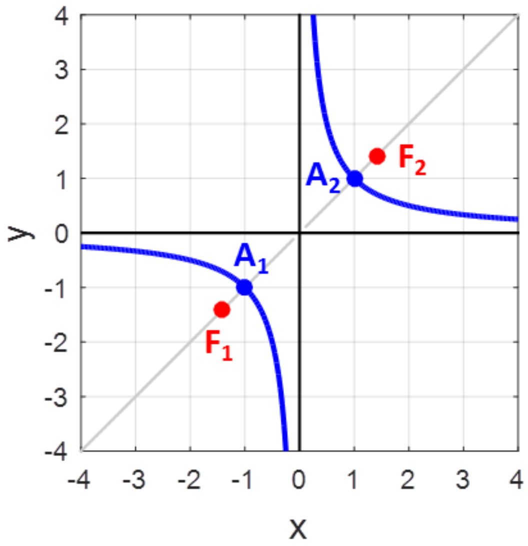


Example: Verify the information shown in the figure below ($x\,y=1 \quad y=1/x \quad a=b=\sqrt{2}$)



The vertices A_1 and A_2 can be found from the solution of the equations

$$y=1/x \quad \text{and} \quad y=x \quad \Rightarrow \quad x=1 \quad y=1 \quad \text{and} \quad x=-1 \quad y=-1$$

The Cartesian coordinates are $A_1(-1, -1)$ and $A_2(1,1)$

The parameter a is equal to the distance OA_2 $\quad a=\sqrt{1^2+1^2}=\sqrt{2}$

For a rectangular hyperbola $\quad a=b=\sqrt{2} \quad c^2=a^2+b^2 \quad \Rightarrow \quad c=2$

The focal length is c = 2 (distance F_1 = F_2 = 2), therefore, the Cartesian coordinates of F_1 and F_2 are

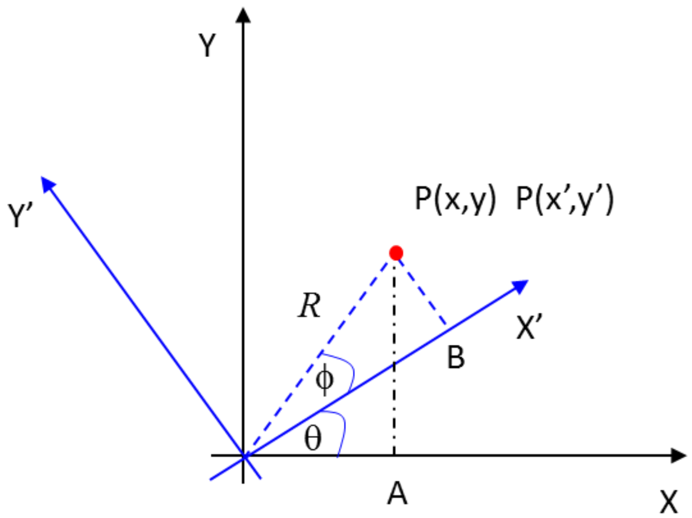
$$F_1(-\sqrt{2}, -\sqrt{2}) \text{ and } F_2(\sqrt{2}, \sqrt{2})$$

The eccentricity e is $e = \frac{c}{a} = \frac{2}{\sqrt{2}} = \sqrt{2}$

ROTATION OF AXES

The equation for the rectangular hyperbola $x\,y = a^2 / 2$ is the hyperbola $x^2 - y^2 = a^2$ referred to an XY coordinate system that has been rotated anticlockwise through an angle of 45°.

Suppose that a set of XY-coordinate axes has been rotated about the origin by an angle θ, where 0 < θ < π/2, to form a new set of X'Y' axes. We would like to determine the coordinates for a point P in the plane relative to the two coordinate systems.



From the two right angle triangles shown in the figure, we can give the coordinates of the point P in Cartesian and polar coordinates for both sets of axes.

$$P(x, y)$$

$$x = R\cos(\theta + \phi) = R\cos\theta\cos\phi - R\sin\theta\sin\phi$$

$$y = R\sin(\theta + \phi) = R\sin\theta\cos\phi + R\cos\theta\sin\phi$$

$$P(x', y')$$

$$x' = R\cos(\phi)$$

$$y' = R\sin(\phi)$$

$$x = x'\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\cos\theta$$

$$x\cos\theta = x'\cos^2\theta - y'\sin\theta\cos\theta$$

$$y\sin\theta = x'\sin^2\theta + y'\sin\theta\cos\theta$$

$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

Coordinate Rotation Formulas If a rectangular XY coordinate system is rotated through an angle θ to form an X'Y' coordinate system, then a point P(x, y) will have coordinates P(x', y') in the new system, where (x, y) and (x', y') are related by

$$x = x'\cos\theta - y'\sin\theta \qquad y = x'\sin\theta + y'\cos\theta$$

$$x' = x\cos\theta + y\sin\theta \qquad y' = -x\sin\theta + y\cos\theta$$

Example

Show that the graph of the equation $xy = a^2/2$ is a hyperbola by rotating the XY axes through an angle of π/4 rad (45°).

Solution

Denoting a point in the rotated system by (x', y'), we have

$$x = x'\cos\theta - y'\sin\theta \qquad y = x'\sin\theta + y'\cos\theta$$

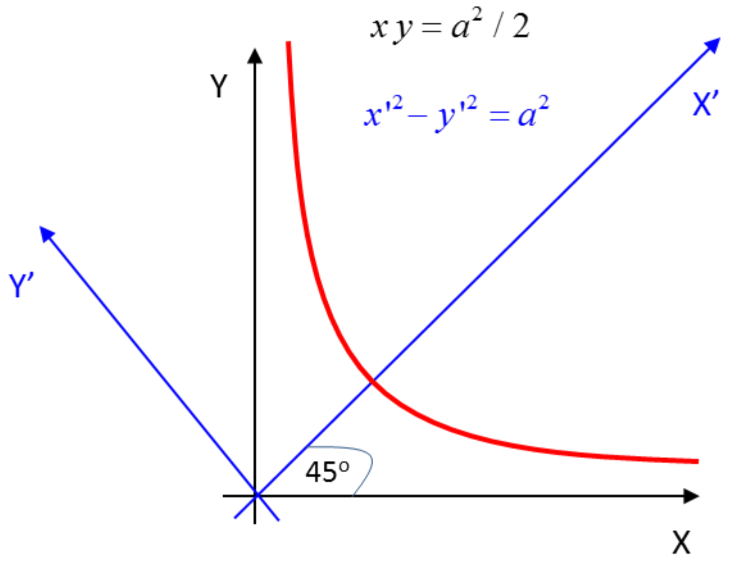
$$\theta = \pi/4 \text{ rad} \quad \sin\theta = 1/\sqrt{2} \quad \cos\theta = 1/\sqrt{2}$$

$$xy = \left(\frac{1}{\sqrt{2}}\right)(x' - y')\left(\frac{1}{\sqrt{2}}\right)(x' + y')$$

$$xy = \frac{1}{2}\left(x'^2 - y'^2\right) = \frac{a^2}{2}$$

$$x'^2 - y'^2 = a^2$$

In the X'Y' coordinate system, then, we have a standard position hyperbola whose asymptotes are $y' = \pm x'$.



The constant a is the distance from the origin O(0, 0) to one of the vertices (A_1 or A_2) of the hyperbola.

The constant c is the distance from the origin O(0, 0) to one of the focal points (F_1 or F_2).

The constant d is the length of the perpendicular line joining a point (D_1 or D_2) on one of the directrices to the origin O(0, 0).

The transformation of points and lines between the X'Y' and XY Cartesian coordinate systems is done by using the relationships

$$\theta = \pi / 4 \text{ rad} = 45^o$$

$$x = \frac{1}{\sqrt{2}}(x' - y') \qquad y = \frac{1}{\sqrt{2}}(x' + y')$$

$$x' = \frac{1}{\sqrt{2}}(x + y) \qquad y' = \frac{1}{\sqrt{2}}(-x + y)$$

Vertex A_2

X'Y' axes $A_2(a, 0)$ $x' = a$ $y' = 0$

XY axes $A_2\left(a/\sqrt{2}, a/\sqrt{2}\right)$ $x = a/\sqrt{2}$ $y = a/\sqrt{2}$

Focal Point F_2 $c = \sqrt{2}\,a$

X'Y' axes $F_2(\sqrt{2}\,a, 0)$ $x' = \sqrt{2}\,a$ $y' = 0$

XY axes $F_2(a, a)$ $x = a$ $y = a$

Point D_2 on directrix $d = a/\sqrt{2}$

X'Y' axes $D_2\left(a/\sqrt{2}, 0\right)$ $x' = a/\sqrt{2}$ $y' = 0$

XY axes $D_2(a/2, a/2)$ $x = a/2$ $y = a/2$

Asymptotes

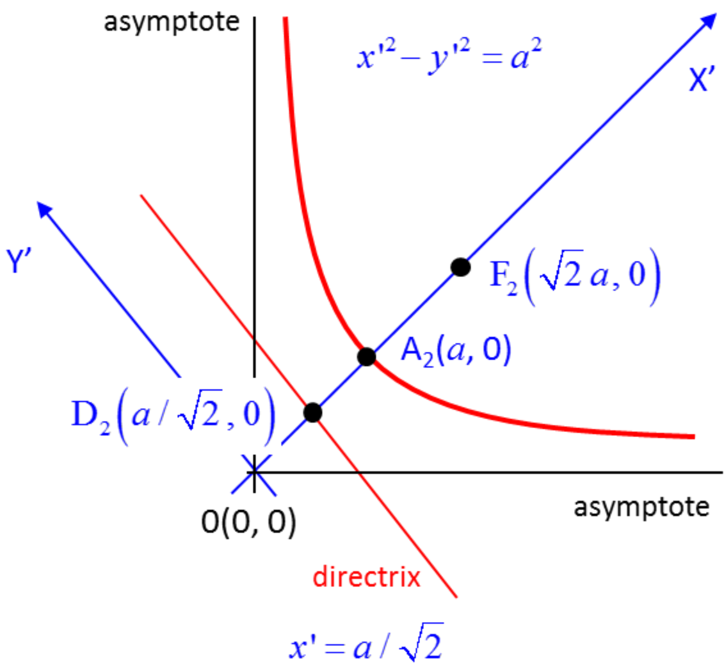
X'Y' axes $y' = x'$ $y' = -x'$

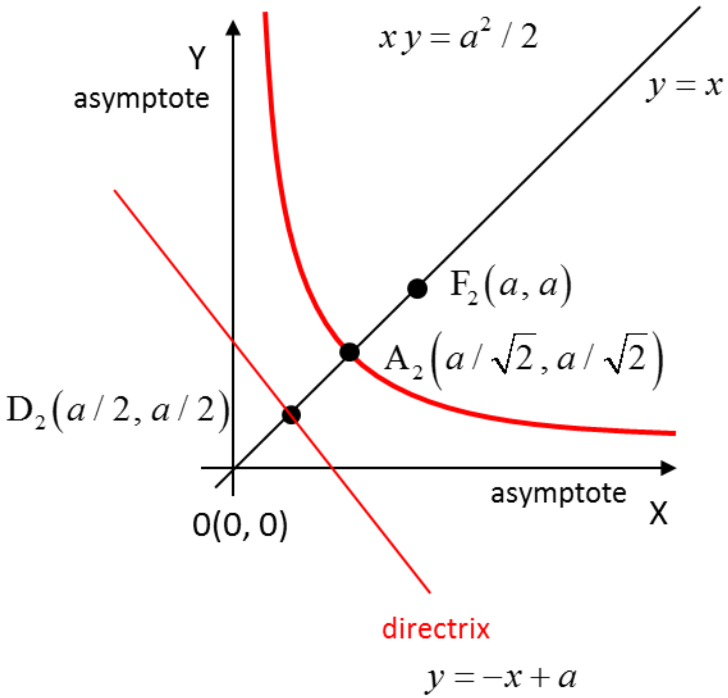
XY axes $x = 0$ $y = 0$

Directrices

X'Y' axes $x' = -a/2$ $y' = 0$ $x' = a/2$ $y' = 0$

XY axes $y = -x + a$ $y = -x - a$





Parametric equation for a rectangular hyperbola

The equation for the rectangular hyperbola is

$$xy = \frac{a^2}{2}$$

where a is the distance from the origin to a vertex. This equation can be expressed in parametric coordinates $\left(kt, \frac{k}{t}\right)$ where k is a constant and t is a variable parameter. For a point on the hyperbola

$$xy = (kt)\left(\frac{k}{t}\right) = k^2$$

Hence $\quad xy = k^2 = \dfrac{a^2}{2} \qquad k = \dfrac{a}{\sqrt{2}} \qquad a = \sqrt{2}\,k$

The focal length c (distance from the origin to a focal point) is

$$c = \sqrt{2}\,a = 2k \qquad k = \frac{c}{2}$$

*** In these notes c is used exclusively to represent the focal length. However, the syllabus and in exam questions, unfortunately in some instances c is used as the focal length and at other times it is used as an arbitrary constant. The syllabus expresses the equation for the rectangular hyperbola in parametric form as $\left(ct, \frac{c}{t}\right)$ but c is just a constant and not the focal length. In my notes, I will use k for the constant and c to be the focal length. This is a much better approach.