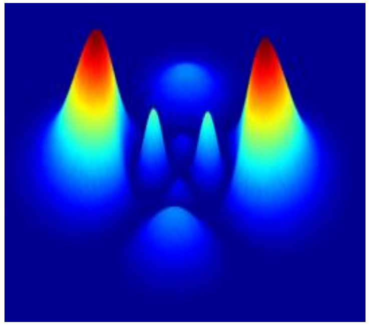
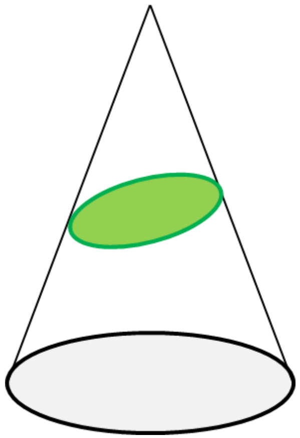


ADVANCED HIGH SCHOOL MATHEMATICS

CONICS

ELLIPSE

Ellipses are the **closed** type of **conic sections**: a plane curve that results from the intersection of a cone by a plane. Ellipses have many similarities with the other two forms of conic sections: the **parabolas** and the **hyperbolas**, both of which are open and unbounded. The cross section of a cylinder is an ellipse, unless the section is parallel to the axis of the cylinder.



An ellipse obtained as the intersection of a cone with an inclined plane.

An **ellipse** can be defined as the locus of all points that satisfy the equation

$$\frac{(x-x_1)^2}{a^2}+\frac{(y-y_1)^2}{b^2}=1$$

Variables: (x, y) the coordinates of any point on the ellipse

Constants: (x_1, x_2) the coordinates of the ellipse's centre

a, b are the radius on the X and Y axes respectively

The **circle** is simply an ellipse where $a = b$. The equation of a circle of radius a and centre (x_1, x_2) is

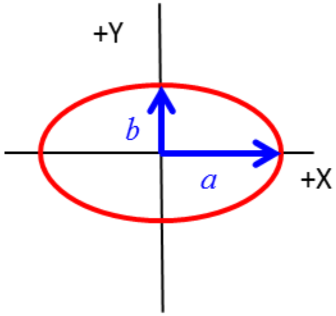
$$(x-x_1)^2+(y-y_1)^2=a^2$$

The equations for an ellipse and circle centred on the **origin** (0, 0) are

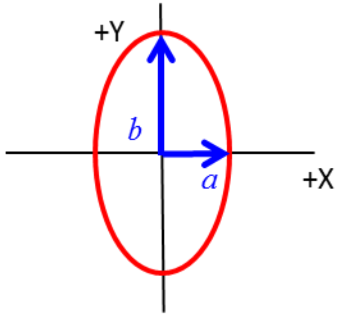
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$ **ellipse**

$$x^2+y^2=a^2$$ **circle**

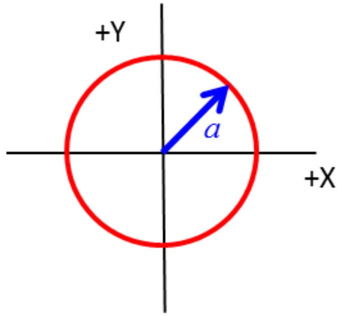
For the ellipse, ***a*** is best refer to as the **X radius** and ***b*** as the **Y radius**. When $a > b$, a is often called the **semi-major axis** and b is called the **semi-minor axis**.



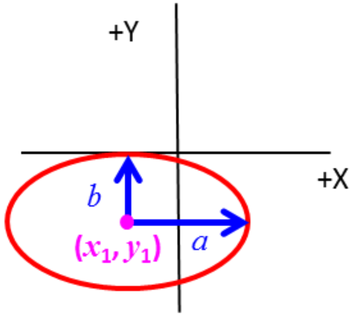
$a > b$



$a < b$



$a = b$

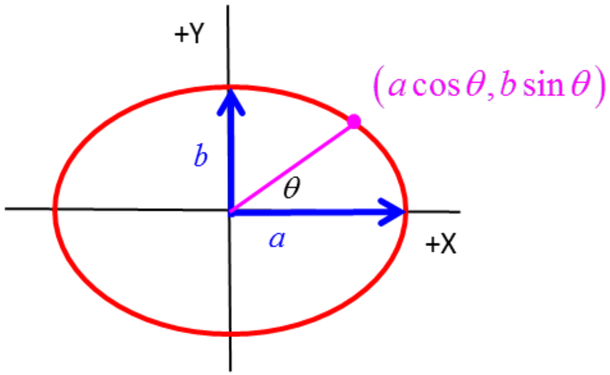


As seen from the equation and graph of an ellipse it is a multi-valued function. For each value of x there are two y values.

The equation for an ellipse with can also be given in **parametric form**

$$x = x_1 + a\cos(\theta)$$
$$y = y_1 + b\sin(\theta)$$

where θ is an angle which ranges from 0 to 2π radians.



$a > b$

The equation of a circle in parametric form with the centre at the origin (0,0) and radius a is

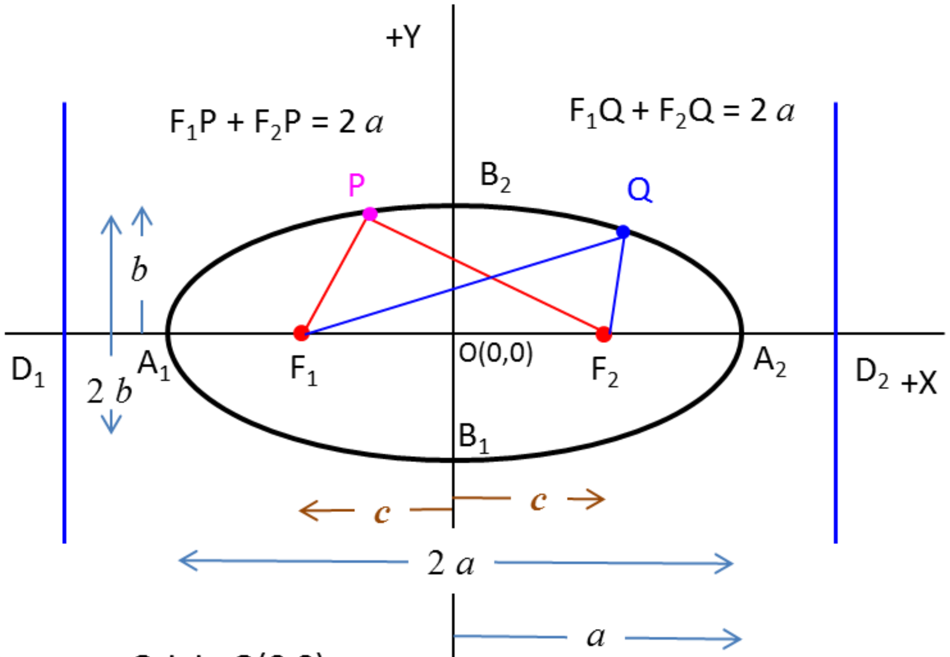
$$x = a\cos(\theta)$$
$$y = a\sin(\theta)$$

$$x = a\cos(\theta) \quad \cos^2(\theta) = x^2/a^2$$
$$y = b\sin(\theta) \quad \sin^2(\theta) = y^2/b^2$$
$$\sin^2(\theta) + \cos^2(\theta) = x^2/a^2 + y^2/b^2 = 1$$

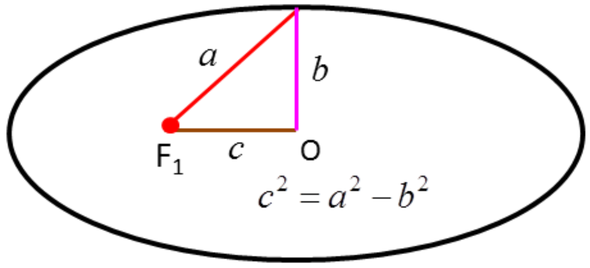
Do the online simulation activity in which you can view the shape an ellipse by changing the values of a and b. Predict the changes in the shape of the ellipse. Then test your predictions and account for any discrepancies between your predictions and observations.

View the online SIMULATION ACTIVITY

Ellipses are common in physics, astronomy and engineering. For example, the orbit of each planet in the solar system is an ellipse with Sun at one of the focal points. The same is true for moons orbiting planets and all other systems having two astronomical bodies.



eccentricity e $\boxed{0 \le e < 1}$



$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

$$c = e\,a \qquad a \ge b$$

$$a = b \Rightarrow e = 0 \Rightarrow \text{circle}$$

$$a \to \infty \Rightarrow e \to 1$$

Centre at Origin O(0,0)

Vertices A_1 ($-a$, 0) and A_2 (a,0)

Vertices B_1 (0,$-b$) and B_2 (0,b)

Foci F_1 ($-c$,0) and F_2 (c,0)

distance $A_1A_2 = 2a$

distance $B_1B_2 = 2b$

distance $F_1P + F_2P = 2a$

distance $F_1Q + F_2Q = 2a$

Directrices (vertical lines)

D_1 ($-a/e$) D_2 (a/e)

$a > b$

X-axis: major axis

Y-axis: minor axis

semi-major axis a

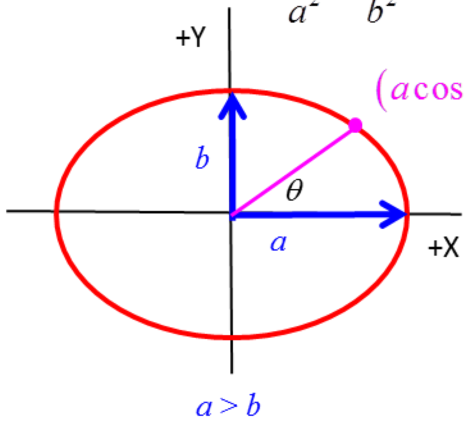
semi-minor axis b

focal length $c^2 = a^2 - b^2$

eccentricity $e = c / a$

Equation of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad x = a\cos(\theta) \quad y = b\sin(\theta)$$



Ellipse is symmetrical about the X-axis and Y-axis

An ellipse may also be defined as the locus of points, the sum of whose distances from two fixed points F_1 and F_2 (the **focal points** or **foci**) is constant and equal to $2a$.

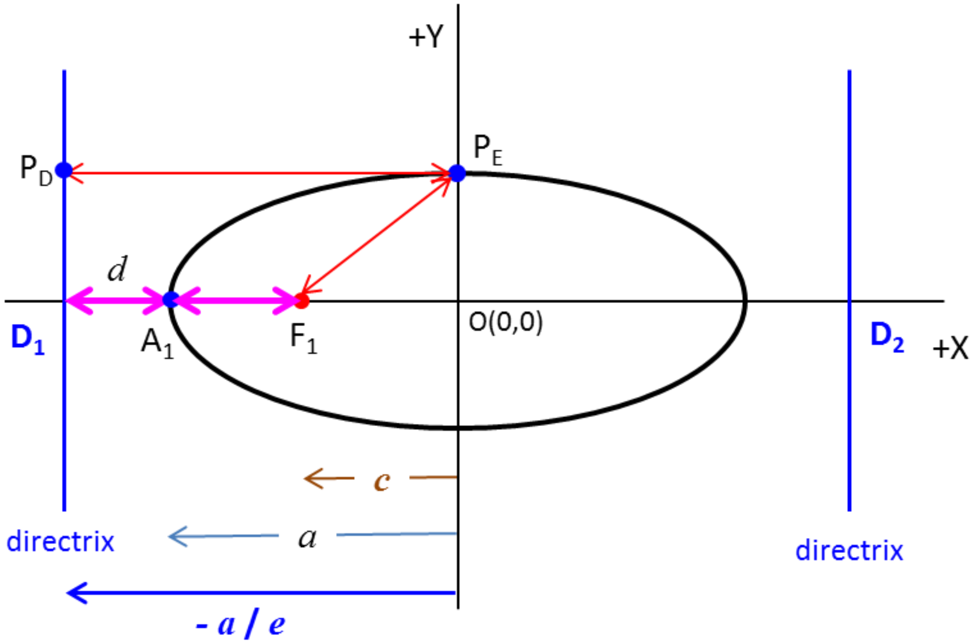
For any point P on the ellipse: $F_1P + F_2P = 2\,a$

The circle is a special type of an ellipse that has both focal points at the same location. The shape of an ellipse (how 'elongated' it is) is represented by its **eccentricity** e, which for an ellipse can be any number from **0** (the limiting case of a circle) to arbitrarily close to but less than **1**.

semi-major axis a semi-minor axis b focal length c **eccentricity** e

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \left(\frac{b}{a}\right)^2} \quad \Rightarrow \quad 0 \le e < 1$$

An ellipse can also be defined as the set of points such that the ratio of the distance of each point on the curve from a given point (called a focus or focal point) to the distance from that same point on the curve to a given line (called the **directrix**) is a constant, called the **eccentricity** *e* of the ellipse.



distances	ratios
$OF_1 = c$	$F_1A_1 / D_1A_1 = (a - c) / d$
$OA_1 = a$	$F_1P_E / P_DP_E = a / (a + d)$
$F_1P_E = a$	Ellipse
$F_1A_1 = a - c$	$F_1A_1 / D_1A_1 = F_1P_E / P_DP_E$
$D_1A_1 = d$	$(a - c) / d \; = \; = a / (a + d)$
$P_DP_E = d + a$	$d = a^2 / c - a$
	$a + d = a^2 / c \quad e = c / a$
	$a + d = a / e$
	ratio $\mathbf{F_1P_E / P_DP_E = e}$
	$\mathbf{D_1\ (- a / e, 0)}$
	$\mathbf{D_2\ (+ a / e, 0)}$

Derivation of the Cartesian form of an ellipse from the focus-directrix definition

$$d_1 / d_2 = e \quad d_1{}^2 = e^2 d_2{}^2$$

$$d_1{}^2 = (x + c)^2 + y^2 \qquad d_2{}^2 = (x + a/e)^2$$

$$c = e\,a \quad e = c/a \quad c^2 = a^2 - b^2$$

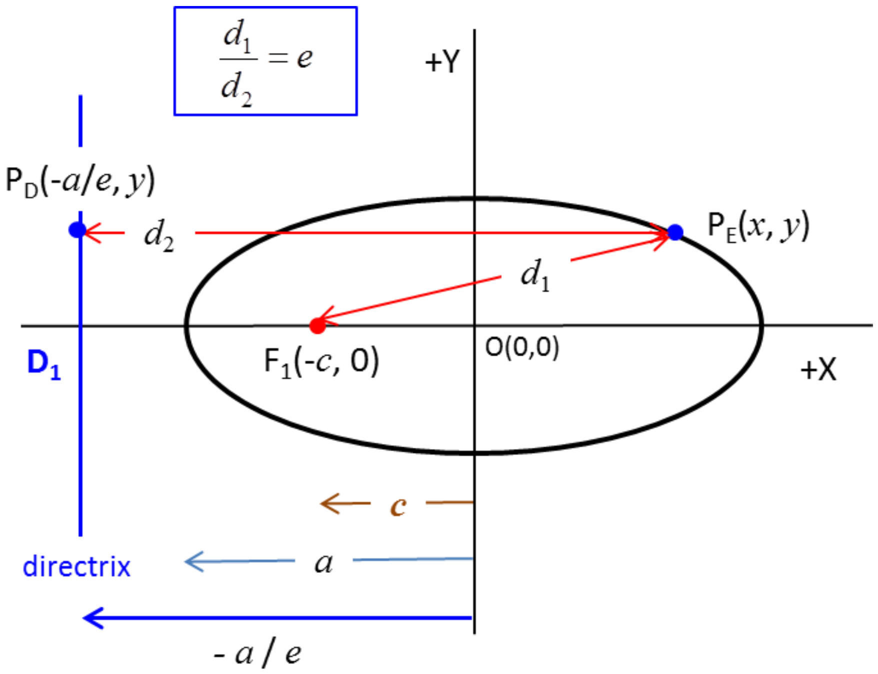
$$e^2 = \frac{a^2 - b^2}{a^2} \quad 1 - e^2 = b^2 / a^2$$

$$(x + e\,a)^2 + y^2 = e^2\,(x + a/e)^2$$

$$y^2 = x^2\left(e^2 - 1\right) + x\left(2a\,e - 2a\,e\right) + a^2\left(1 - e^2\right)$$

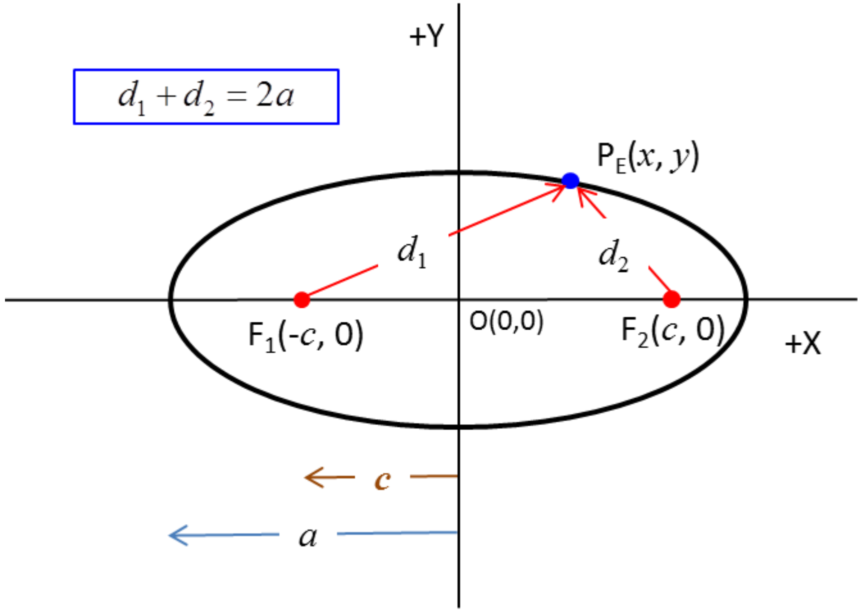
$$y^2 = -x^2\left(b^2 / a^2\right) + b^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad QED$$



Derivation of the Cartesian form for an ellipse from the locus of points, the sum of whose distances from two the two focal points is constant and equal to $2a$.

You solve the problem by calculating the distances d_1 and d_2 as in the above example. However, the algebra becomes "very messy". For interest only, you can view the full derivation on wikipedia.



https://en.wikipedia.org/wiki/Derivation_of_the_Cartesian_form_for_an_ellipse

Equation of tangents to an ellipse

The equation of an ellipse with its centre at the origin is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The equation of the **tangent** to the ellipse can be expressed as

$$y = M_1 x + B_1$$

where M_1 is the gradient and B_1 is the intercept of the straight line.

The first derivative dy/dx at a point P$\left(x_P, y_P\right)$ on the ellipse gives the gradient of the tangent to the ellipse at that point. The implicit differentiation of the equation for the ellipse gives

$$\frac{2\,x}{a^2}+\frac{2\,y}{b^2}\frac{dy}{dx}=0 \quad \Rightarrow \quad \frac{dy}{dx}=-\left(\frac{b}{a}\right)^2\frac{x}{y}$$

The gradient M_1 of the tangent at the point P$\left(x_P, y_P\right)$ is

$$M_1=-\left(\frac{b}{a}\right)^2\frac{x_P}{y_P}$$

and the intercept B_1 is

$$B_1 = y_P + \left(\frac{b}{a}\right)^2\left(\frac{{x_P}^2}{y_P}\right)$$

The equation of the tangent is

$$y = M_1 x + B_1$$

$$M_1 = -\left(\frac{b}{a}\right)^2 \frac{x_P}{y_P}$$

$$B_1 = y_P + \left(\frac{b}{a}\right)^2 \left(\frac{{x_P}^2}{y_P}\right)$$

$$y = -\left(\frac{b}{a}\right)^2 \left(\frac{{x_P}^2}{y_P}\right) x + y_P + \left(\frac{b}{a}\right)^2 \left(\frac{{x_P}^2}{y_P}\right)$$

$$\left(\frac{y_P}{b^2}\right) y = -\left(\frac{x_P}{a^2}\right) x + \left(\frac{{y_P}^2}{b^2}\right) + \left(\frac{{x_P}^2}{a^2}\right) \qquad \left(\frac{{x_P}^2}{a^2}\right) + \left(\frac{{y_P}^2}{b^2}\right) = 1$$

$$\left(\frac{x_P}{a^2}\right) x + \left(\frac{y_P}{b^2}\right) y = 1$$

The tangent crosses the X-axis (y = 0) at the point T$\left(x_T, 0\right)$ where

$$x_T = \frac{a^2}{x_P}$$

Equation of a normal to an ellipse

The normal to an ellipse at the point P(x_P, y_P) is a straight line that is perpendicular to the tangent at that point.

Equation of tangent $\quad y = M_1\, x + B_1$

Equation of normal $\quad y = M_2\, x + B_2$

Gradient of the normal M_2

$$M_2 = \frac{-1}{M_1} = \left(\frac{a}{b}\right)^2 \frac{y_P}{x_P}$$

Intercept of normal B_2

$$B_2 = y_P - \left(\frac{a}{b}\right)^2 y_P = y_P \left(\frac{b^2 - a^2}{b^2}\right)$$

The equation of the normal is

$$y = M_2\, x + B_2$$

$$M_2 = \left(\frac{a}{b}\right)^2 \frac{y_P}{x_P}$$

$$B_2 = y_P\left(\frac{b^2 - a^2}{b^2}\right)$$

$$y = \left(\frac{a}{b}\right)^2 \frac{y_P}{x_P}\, x + y_P\left(\frac{b^2 - a^2}{b^2}\right)$$

$$\left(\frac{a^2}{x_P}\right) x - \left(\frac{b^2}{y_P}\right) y = a^2 - b^2$$

The normal crosses the X-axis ($y = 0$) at the point N$\left(x_N, 0\right)$where

$$x_N = \left(\frac{a^2 - b^2}{a^2}\right) x_P$$

No is no need to remember the formulae for the tangent and normal to the ellipse. It is best to derive the equations from first principles.

Examples

Consider the ellipse $9x^2 + 25y^2 - 225 = 0$

Verify all the numerical values and information in the following two figures for two points on the ellipse at $x = 3.5$ and $x = -3.8$.

Note: the **reflection property of an ellipse** – the normal to an ellipse at a point P on it is equally inclined to the focal chords through P $\theta_1 = \theta_2$

$P(x, y) = (3.5, 2.1424)$

semi-major axis $a = 5$ major axis $2a = 10$

semi-minor axis $b = 3$ minor axis $2b = 6$

focal length $[OF_1 \; OF_2]$ $c = 4$ eccentricity $e = 0.8$

directrices 1: $x = -6.25$ directrices 2: $x = 6.25$

slope tangent $M_1 = -0.588$ slope normal $M_2 = 1.7$

intercept tangent $B_1 = 4.2$ intercept normal $B_2 = -3.81$

T tangent cross X-axis: $x = 7.14$ N normal cross X-axis: $x = 2.24$

distances: $PF_1 = 7.8$ $PF_2 = 2.2$ $PF_1 + PF_2 = 10$

distances: $PF_1 = 7.8$ $PD_1 = 9.75$ $PF_1 / PD_1 = 0.8$

distances: $PF_2 = 2.2$ $PD_2 = 2.75$ $PF_2 / PD_2 = 0.8$

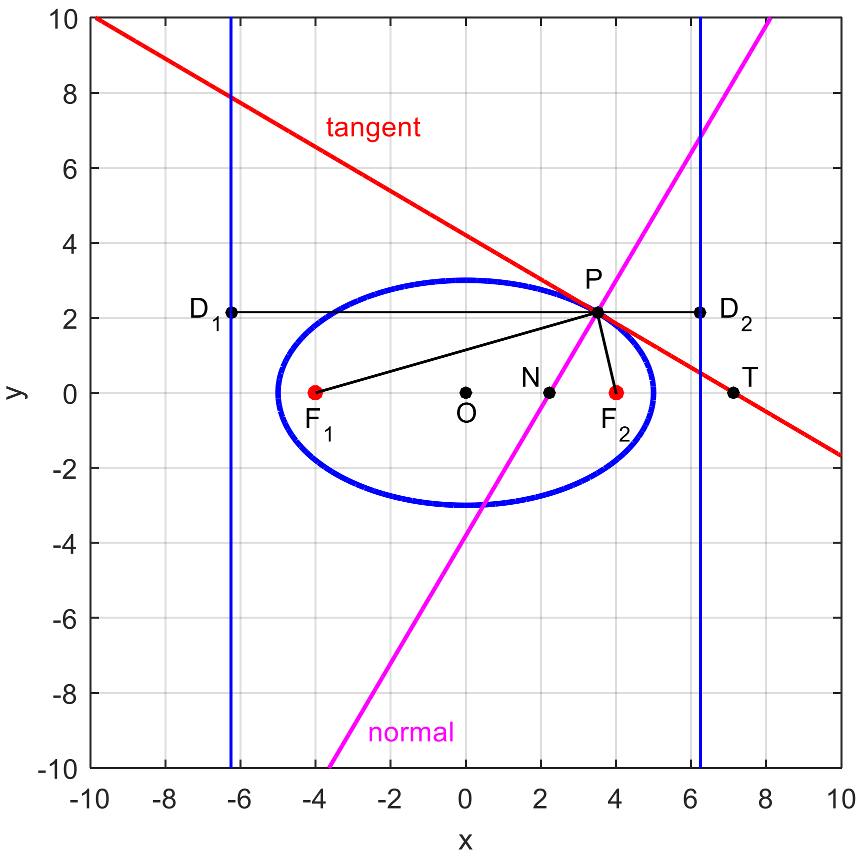
angles: $\theta_1 = NPF_1$ $\theta_2 = NPF_2$

angles: $\theta_3 = NF_1P$ $\theta_4 = PNF_2$ $\theta_5 = PF_2T$

angles: $\theta_1 = \theta_4 - \theta_3$ $\theta_2 = \theta_5 - \theta_4$

angle [degrees] $\theta_4 = 59.5$ $\theta_3 = 15.9$ $\theta_1 = 43.6$

angle [degrees] $\theta_4 = 59.5$ $\theta_5 = 103$ $\theta_2 = 43.6$



$P(x, y) = (-3.8, 1.9498)$

semi-major axis $a = 5$ | major axis $2a = 10$

semi-minor axis $b = 3$ | minor axis $2b = 6$

focal length $[OF_1\ OF_2]$ $c = 4$ | eccentricity $e = 0.8$

directrices 1: $x = -6.25$ | directrices 2: $x = 6.25$

slope tangent $M_1 = 0.702$ | slope normal $M_2 = -1.43$

intercept tangent $B_1 = 4.62$ | intercept normal $B_2 = -3.47$

T tangent cross X-axis: $x = -6.58$ | N normal cross X-axis: $x = -2.43$

distances: $PF_1 = 1.96$ $PF_2 = 8.04$ $PF_1 + PF_2 = 10$

distances: $PF_1 = 1.96$ $PD_1 = 2.45$ $PF_1 / PD_1 = 0.8$

distances: $PF_2 = 8.04$ $PD_2 = 10.1$ $PF_2 / PD_2 = 0.8$

angles: $\theta_1 = NPF_1$ $\theta_2 = NPF_2$

angles: $\theta_3 = NF_1P$ $\theta_4 = PNF_2$ $\theta_5 = PF_2T$

angles: $\theta_1 = \theta_4 - \theta_3$ $\theta_2 = \theta_5 - \theta_4$

angle [degrees] $\theta_4 = 125$ $\theta_3 = 84.1$ $\theta_1 = 40.9$

angle [degrees] $\theta_4 = 125$ $\theta_5 = 166$ $\theta_2 = 40.9$