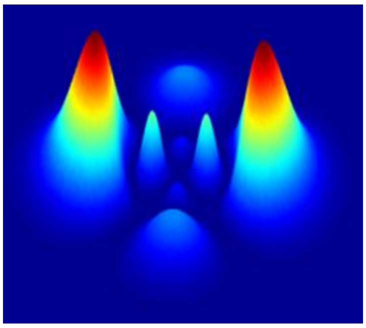


ADVANCED HIGH SCHOOL MATHEMATICS

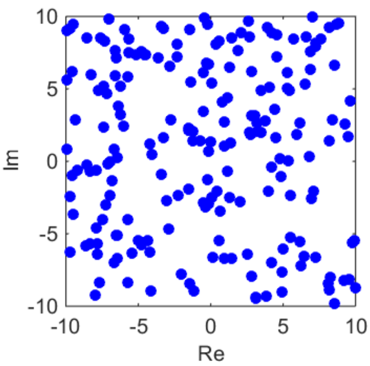
COMPLEX NUMBERS

CURVES AND REGIONS

CURVES or LOCI

A **locus** (plural: **loci**) is a set of points whose location satisfies or is determined by one or more specified conditions.

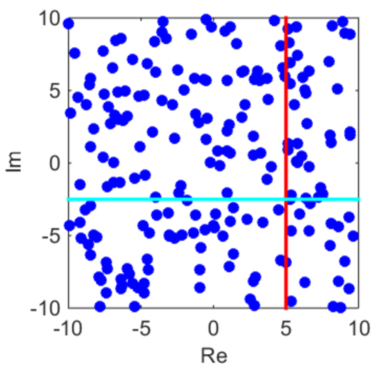
Lines



The complex number z

$$z = x + i\ y$$

corresponds to the point (x,y) in an Argand diagram. In the figure shown, each blue dot represents a complex number with coordinates (x, y).



Now consider only those complex numbers where

$$z = 5 + i\ y \qquad \mathrm{Re}(z) = 5$$

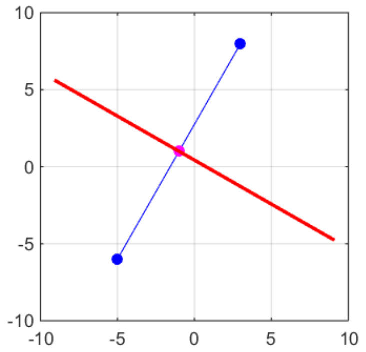
This set of complex numbers can only be located on the vertical line $x = 5$ on the Argand Diagam.

If the complex numbers are of the form

$z = x + i\left(-2.5\right)y \quad \mathrm{Im}(z) = -2.5$ then this set of complex numbers will lie on the horizontal line **$y = -2.5$**.

Let z_1 and z_2 be two points on an Argand diagram. Consider an equation of the form

$$\left|z - z_1\right| = \left|z - z_2\right|$$



The distance between the points z and z_1 is $d_1 = \left|z - z_1\right|$ and the distance between the points z and z_2 is $d_2 = \left|z - z_2\right|$. So for any z value we must have $d_1 = d_2$. Therefore z must correspond to a **straight line** passing through the centre of the line joining the two points z_1 and z_2 and perpendicular to it.

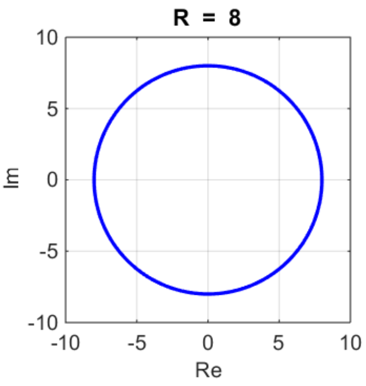
$\left|z - z_1\right| = \left|z - z_2\right|$ equation of the perpendicular bisector of the line joining z_1 and z_2.

Circles

$$z = x + i\ y$$

The equation of a circle of radius R with centre (0, 0) is

$$\left|z\right| = R \qquad \left|z\right| = \sqrt{x^2 + y^2} = R \qquad x^2 + y^2 = R^2$$

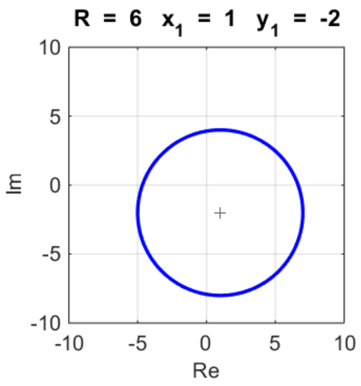


The equation of a circle of radius R with centre given by z_1 (x_1, y_1) is

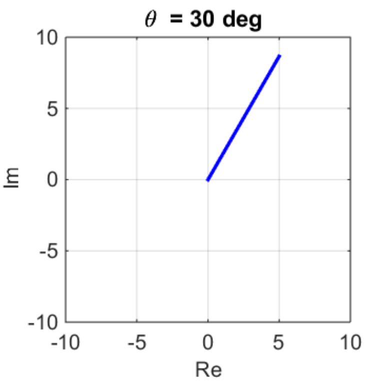
$$\left|z - z_1\right| = R$$

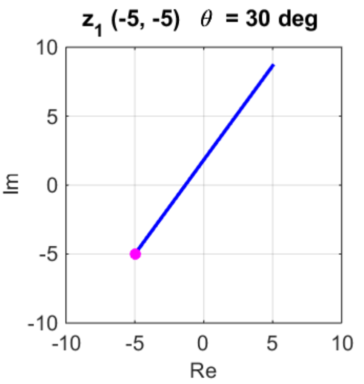
$$\left|z\right| = \sqrt{\left(x - x_1\right)^2 + \left(y_1\right)^2} = R$$

$$\left(x - x_1\right)^2 + \left(y - y_1\right)^2 = R^2$$



Arguments





$$z = x + i\ y \qquad \arg(z) = \theta = a\tan\left(\frac{y}{x}\right)$$

$$z_1 = x_1 + i\ y_1$$

The equation $\theta = \arg(z)$ corresponds to the line draw from the origin (0, 0) to any complex numbers z such that the angle of the line with respect to the real axis is θ.

The equation $\theta = \arg(z - z_1)$ corresponds to the line draw from the point (x_1, y_1) to any complex numbers z such that the angle of the line with respect to the real axis is θ.

Locus of an arc

The locus of a point z on an Argand diagram that satisfies the relationship

$$Arg\left(\frac{z - z_1}{z - z_2}\right) = Arg\left(z - z_1\right) - Arg\left(z - z_1\right) = \alpha$$

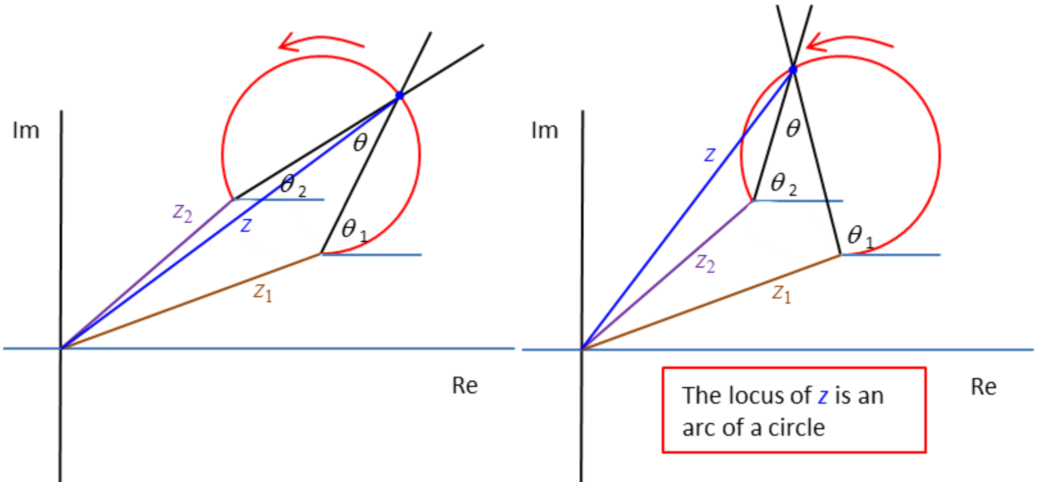
is the **arc of a circle**.

Let $\theta_1 = Arg\left(z - z_1\right) \quad \theta_2 = Arg\left(z - z_2\right)$ then $\theta = \theta_1 - \theta_2$

$\theta_1 = Arg\left(z - z_1\right)$ is the locus of a straight line (1) starting at z_1 and making an angle θ_1 with the real axis.

$\theta_2 = Arg\left(z - z_2\right)$ is the locus of a straight line (2) starting at z_1 and making an angle θ_2 with the real axis.

$Arg\left(\frac{z - z_1}{z - z_2}\right) = Arg\left(z - z_1\right) - Arg\left(z - z_1\right) = \alpha$ is the locus of the point z which is the point of intersection of the two straight lines (1) and (2). As the angle θ_1 increases and θ_2 decreases with θ remaining constant the intersection point z moves along the arc of a circle in an anticlockwise direction (starting at z_1 the arc is in anticlockwise sense to z_2 $z \neq z_1 \quad z \neq z_2$). The points z_1 and z_2 and all the points z lie on the circle. If θ is acute ($\theta < 90^o$) then the points z are on the major arc and if θ is obtuse ($\theta > 90^o$) z is on the minor arc.



$$Arg\left(\frac{z-z_1}{z-z_2}\right) = Arg\left(z-z_1\right) - Arg\left(z-z_2\right) = \theta$$

$$\theta_1 = Arg\left(z-z_1\right) \quad \theta_2 = Arg\left(z-z_2\right)$$

$$\theta = \theta_1 - \theta_2$$

Plot of z as θ_1 increases

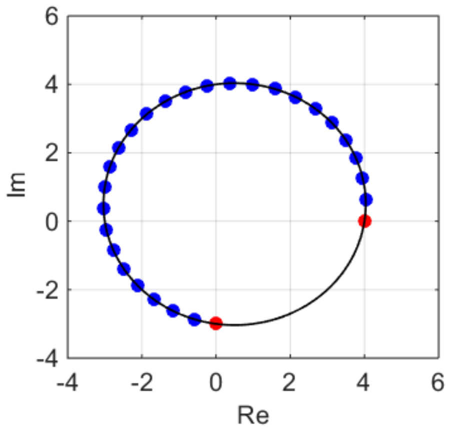
$$Arg\left(\frac{z-4}{z+3i}\right)$$

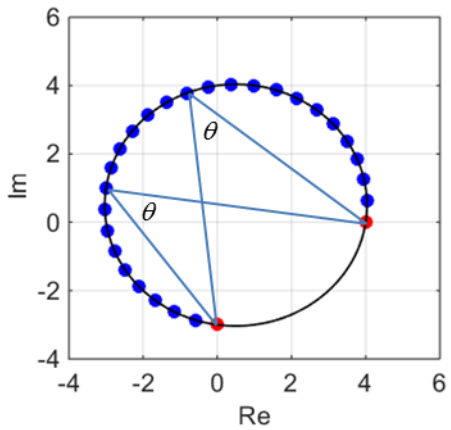
$$= Arg\left(z-4\right) - Arg\left(z+3i\right) = \pi/4$$

$z_1 = 4 \quad z_2 = -3i \quad \theta = \pi/4$

$\theta_1 = 82^\circ$ to 212° steps of 5°

z moves anticlockwise around the major arc as θ_1 increases $\quad z \neq z_1 \quad z \neq z_2$





Plot of z as θ_1 increases

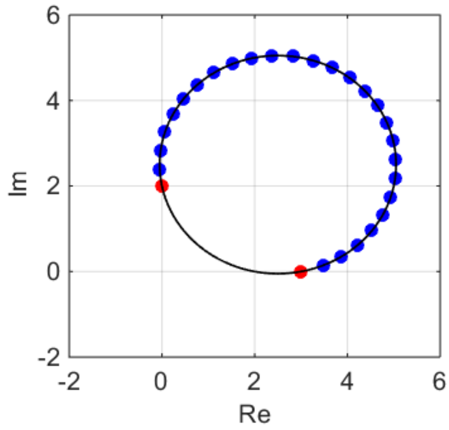
$$Arg\left(\frac{z-3}{z-2i}\right)$$

$$= Arg(z-3) - Arg(z-2i) = \pi/4$$

$z_1 = 3 \quad z_2 = 2i \quad \theta = \pi/4$

$\theta_1 = 17^{\circ}$ to 142° steps of 5°

z moves anticlockwise around the major arc as θ_1 increases $z \neq z_1 \quad z \neq z_2$



Plot of z as θ_1 increases

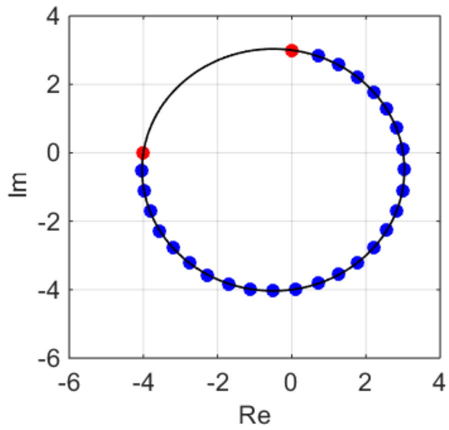
$$Arg\left(\frac{z+4}{z+3i}\right)$$

$$= Arg(z+4) - Arg(z-3i) = \pi/4$$

$z_1 = -4 \quad z_2 = 3i \quad \theta = \pi/4$

$\theta_1 = 86^{\circ}$ to 212° steps of 5°

z moves anticlockwise around the major arc as θ_1 increases $z \neq z_1 \quad z \neq z_2$



Plot of z as θ_1 increases

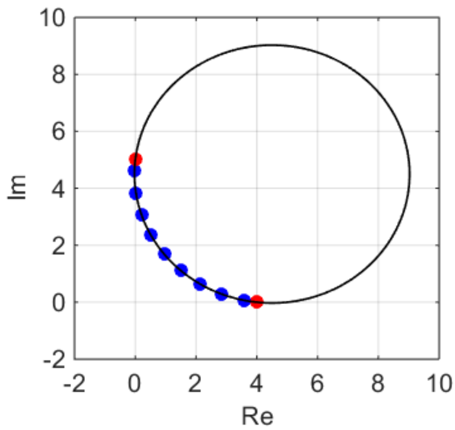
$$Arg\left(\frac{z-5i}{z-4}\right)$$

$$= Arg(z-5i) - Arg(z-4) = 3\pi/4$$

$z_1 = 5i \quad z_2 = 4 \quad \theta = 3\pi/4$

$\theta_1 = 86^{\circ}$ to 126° steps of 5°

z moves anticlockwise around the minor arc as θ_1 increases $z \neq z_1 \quad z \neq z_2$



From the coordinates of the three points z, z_1 and z_2 on the circle you can find the centre of the circle and its radius.

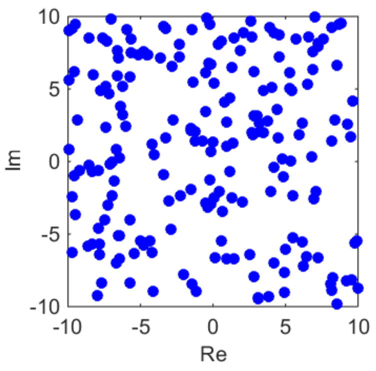
REGIONS

The complex number z

$$z = x + i\ y$$

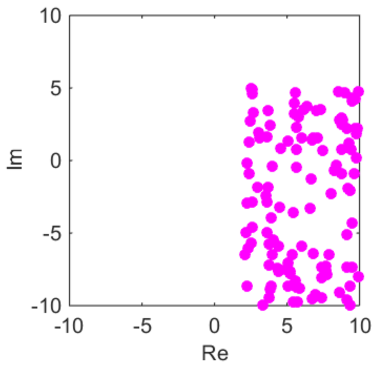
corresponds to the point (x,y) in an Argand diagram.

The figure shows the location of 400 random complex numbers in the complex plane.



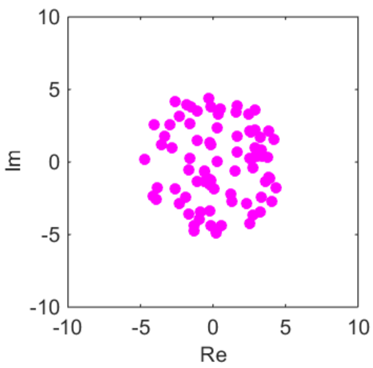
When restrictions are placed upon the values of z, then the plotted values of z that satisfy the restriction will give well defined allowed regions (or lines) in the complex plane.

$$\mathrm{Re}(z) = x > 2 \text{ and } \mathrm{Im}(z) = y < 5$$



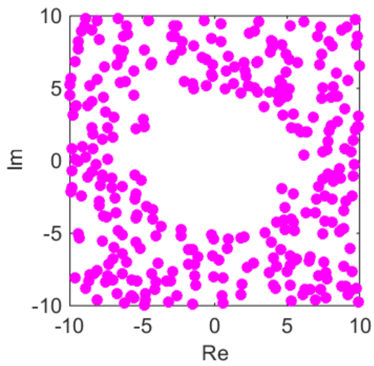
$$|z| = \sqrt{x^2 + y^2} < 5$$

The allowed region is all points within the circle of radius 5 and centre (0, 0) but it does not include points on the circumference.

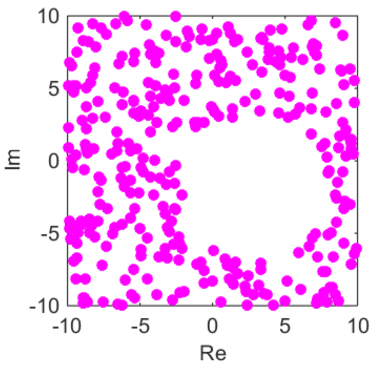


$$|z| = \sqrt{x^2 + y^2} \geq 5$$

The allowed region is all points outside the circle of radius 5 and centre (0, 0) and includes points on the circumference.



$$\left|z-3+2\,i\right|=\sqrt{(x-3)^2+(y+2)^2}\geq 5$$

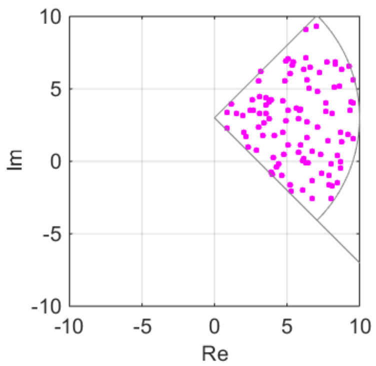


Express the magnitude of the complex number from its rectangular form

$$\left|z-3+2\,i\right|=\sqrt{(x-3)^2+(y+2)^2}\geq 5$$

The allowed region is all points outside the circle of radius 5 and centre (3, -2) and includes points on the circumference.

$$\left|z-3\,i\right|\leq 10 \qquad -\frac{\pi}{4}<Arg\left(z-3\,i\right)<+\frac{\pi}{4}$$



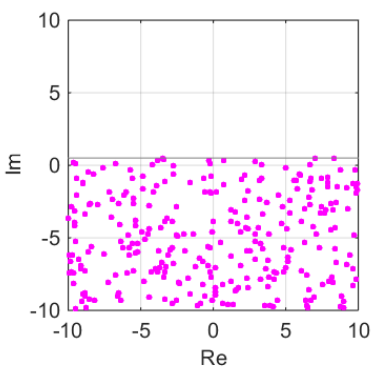
$$z_1=0-3\,i \qquad \left|z-3\,i\right|\leq 10 \quad \Rightarrow$$

allowed region for *z* is all points inside the circle of centre (0,3) and radius 10.

$$-\frac{\pi}{4}<Arg\left(z-3\,i\right)<+\frac{\pi}{4} \quad \Rightarrow$$

Starting at the point (0, 3), the allowed region for *z* is the wedge between the angles -π/4 and +π/4 as measured from a horizontal line (parallel to the real axis) through (0, 3).

$$\left|z\right|\leq\left|z-i\right|$$



Express the complex numbers in rectangular form and then rearrange the inequality

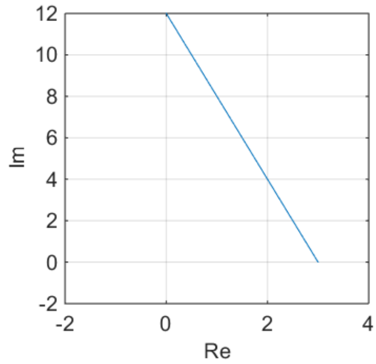
$$x^2+y^2\leq x^2+(y-1)^2 \quad \Rightarrow y\leq\frac{1}{2}$$

The allowed region for the values of *z* is all the points on or below the line given by *y* = ½.

What are the allowed values of z that satisfy the conditions

$$|z-2-3i| = |z-i|$$

Let $z = x + y\,i$



$$|x + y\,i - 2 - 3i| = |x + y\,i - i|$$
$$|(x-2) + i\,(y-3)| = |x + i\,(y-1)|$$
$$(x-2)^2 + (y-3)^2 = x^2 + (y-1)^2$$
$$y = -4\,x + 12$$

Therefore, the allowed values for z must all line on the straight line $y = -4\,x + 12$