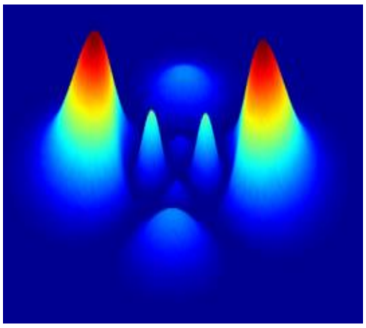

ADVANCED HIGH SCHOOL MATHEMATICS

COMPLEX NUMBERS

CHARACTERISTICS OF COMPLEX NUMBERS

INTRODUCTION

Around the 1900s, one of the greatest technological debates in the U.S.A. was whether the electrical distribution system should be either Dc or AC. Hugh fortunes were to be made or lost by entrepreneurs by investing in either the AC distribution system or the DC system.

The AC system was the winner, in large measure because of the inventive genius of one man: Charles Proteus Steinmetz (1865 – 1923). He was an electrical engineer who migrated to the U.S.A. from Germany. Steinmetz could manipulate vast amounts of mathematics in his mind – he had no need for books of mathematical tables and he could find logarithms to numbers of five significant figures in his head (no scientific calculators in those days). Steinmetz realised that the purely mathematical ideas of **complex variables** could be used to great practical advantage to simplify the mathematical analysis of AC circuits.

> " … electricity from the square root of minus one."
>
> Vladimir Karapetoff (1876 – 1948)

Much mathematics and physics can be solved very easily by using complex variables as it makes the algebra simple and the use of complex variables is absolutely necessary in the study of Quantum Mechanics.

Today, the use of complex variables is an indispensable tool in the modelling of our natural environment and financial markets.

REPRESENTATION OF A COMPLEX VARIABLE

A complex variable makes use of the imaginary number

$$i = \sqrt{-1}$$

A **complex variable** z consists of two parts

Real part x

Imaginary part y

The letter z is the traditional notation for a complex variable in mathematics. Likewise its real part traditionally is called x and its imaginary part y.

Complex variables are written in a number of equivalent representations, each with its own particular usefulness. It is important for you to change from one representation to another since various mathematical operations are performed more easily in one representation than another.

In engineering j is used instead of i as i is used for the current

$$j = \sqrt{-1}$$

Rectangular Form

The **complex variable** z in rectangular form is written as

$$z = x + i\,y$$

The **real part** of the complex variable is

$$\operatorname{Re} z \equiv x$$

The **imaginary part** of the complex variable is

$$\operatorname{Im} z \equiv y$$

- x and y are both real numbers
- The complex nature of z is explicitly and totally contained in the complex number i
- If Im $z = 0$ then $z = x$ is a real number
- If Re $z = 0$ then $z = i\,y$ is a purely imaginary number
- $z = 0$ only if $x = 0$ and $y = 0$
- For complex numbers there is no order relationship or positive or negative character
- The complex number z can also be expressed as an **order pair** (x, y)

ARGAND DIAGRAMS:

Complex numbers in the complex plane

The **complex plane** is a two-dimensional plane that plots the real part of a complex number along the horizontal axis and the imaginary part along the vertical axis.

A **complex number** thus is a point on the complex plane.

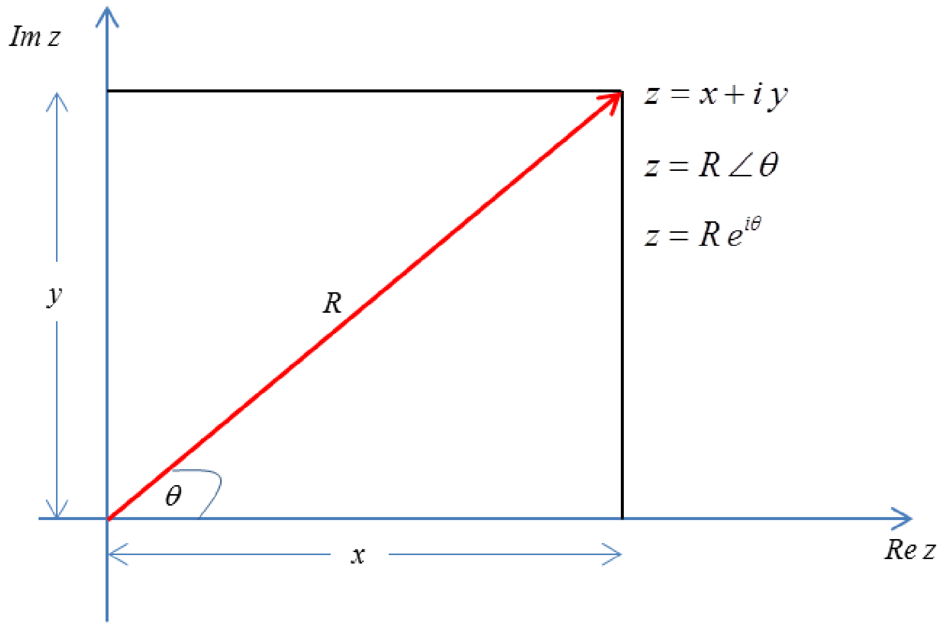

Fig. 1. Argand diagram: Graphical representation of the complex number $z = x + i\,y$ in the complex plane. The complex number *z* is represented as a **vector**.

A complex number is similar to a two-dimensional vector.

By using the Pythagorean theorem, the magnitude of *z* is

$$|z| = R = \sqrt{x^2 + y^2}$$

The **magnitude of the complex number** z is the **positive** square root of the sum of the squares of its real and imaginary parts. The magnitude is always a real number greater than or equal to zero. The magnitude of a complex number is also called its **modulus** or its **absolute value**.

Polar Form

A complex number can also be represented in polar form by specifying its **magnitude** R and the **angle** θ with respect to the real axis in the complex plane. The polar form is written symbolically as

$z = R\angle\theta$ angle measured in radians 2π rad = 360°

From the geometry shown in figure (1)

real part of z $\quad \mathrm{Re}\, z = x = R\cos\theta$

imaginary part of z $\quad \mathrm{Im}\, z = y = R\sin\theta$

modulus of z $\quad |z| = R = \sqrt{x^2 + y^2}$

angle θ $\quad \tan\theta = \dfrac{\mathrm{Im}\, z}{\mathrm{Re}\, z} = \dfrac{y}{x} \qquad \theta = a\tan\left(\dfrac{y}{x}\right) \equiv \tan^{-1}\left(\dfrac{y}{x}\right)$

The angle θ is called the **argument of z** and by convention

$\theta = \mathrm{Arg}(z)$ is restricted to the range $-\pi \le \theta \le +\pi$.

Rectangular Form $\Leftrightarrow$ Polar Form

$$z = R\angle\theta = R\cos\theta + i\, R\sin\theta = R\left(\cos\theta + i\,\sin\theta\right)$$

A short hand way of writing the polar form of a complex number is

$$R\left(\cos\theta + i\,\sin\theta\right) = R\, cis\theta$$

I suggest it is best not to use the cis notation

Exponential Form **VERY USEFUL !!!**

Although this form is not mentioned in the Syllabus, it is the **most useful** form of expressing a complex number and one that you can use to easily solve many problems involving complex numbers that would be more difficult using the rectangular form or the polar form.

The **Euler Identity** states

$$e^{i\theta} = \cos\theta + i\sin\theta$$

[Leonhard Euler (1707 – 1783) – a great mathematician and scientist who gave us many of our modern mathematical notations eg i for $\sqrt{-1}$, e for the base of the natural logarithm and $f(x)$ for a function of x]

Therefore, the exponential form of a complex variable is

$$z = R\,e^{i\theta} \qquad [\theta \text{ in radians}]$$

COMPLEX CONJUGATE

The **complex conjugate** is written as $\bar{z}$ (however, in many physics and maths textbook, the complex conjugate symbol is z^*).

The complex conjugate is found by replacing i with $-i$ wherever it is found.

$$z = x + i\ y \quad \Rightarrow \quad \bar{z} = x - i\ y$$
$$z = x - i\ y \quad \Rightarrow \quad \bar{z} = x + i\ y$$
$$z = R\left(\cos\theta + i\sin\theta\right) \quad \Rightarrow \quad \bar{z} = R\left(\cos\theta - i\sin\theta\right)$$
$$z = R\left(\cos\theta - i\sin\theta\right) \quad \Rightarrow \quad \bar{z} = R\left(\cos\theta + i\sin\theta\right)$$
$$z = R\,e^{i\theta} \quad \Rightarrow \quad \bar{z} = R\,e^{i(-\theta)}$$
$$z = R\,e^{i(-\theta)} \quad \Rightarrow \quad \bar{z} = R\,e^{i\theta}$$

- If a complex number is equal to its complex conjugate then that number is real

$$z = \bar{z} \quad \Rightarrow \quad x + i\ y = x - i\ y \quad \Rightarrow \quad y = 0 \quad \Rightarrow \quad z = \bar{z} = x$$

- If a complex number is equal to negative of its complex conjugate then that purely imaginary

$$z = -\bar{z} \quad \Rightarrow \quad x + i\ y = -\left(x - i\ y\right) \quad \Rightarrow \quad x = 0 \quad \Rightarrow \quad z = \bar{z} = i\ y$$

- The product of the complex number z and its complex conjugate $\bar{z}$ is a real and positive number

$$z\,\bar{z} = (x + i\,y)(x - i\,y)$$
$$= x^2 - i\,x\,y + i\,x\,y + y^2$$
$$= x^2 + y^2 = |z|^2 \qquad i^2 = -1$$

$$z\,\bar{z} = \left(R\,e^{i\theta}\right)\left(R\,e^{-i\theta}\right) = R^2 = |z|^2$$

The product $z\,\bar{z}$ is equal to the square of the magnitude (modulus or absolute value) of the complex number z.

- $\left(z^*\right)^* = (x - i\,y)^* = (x + i\,y) = z$

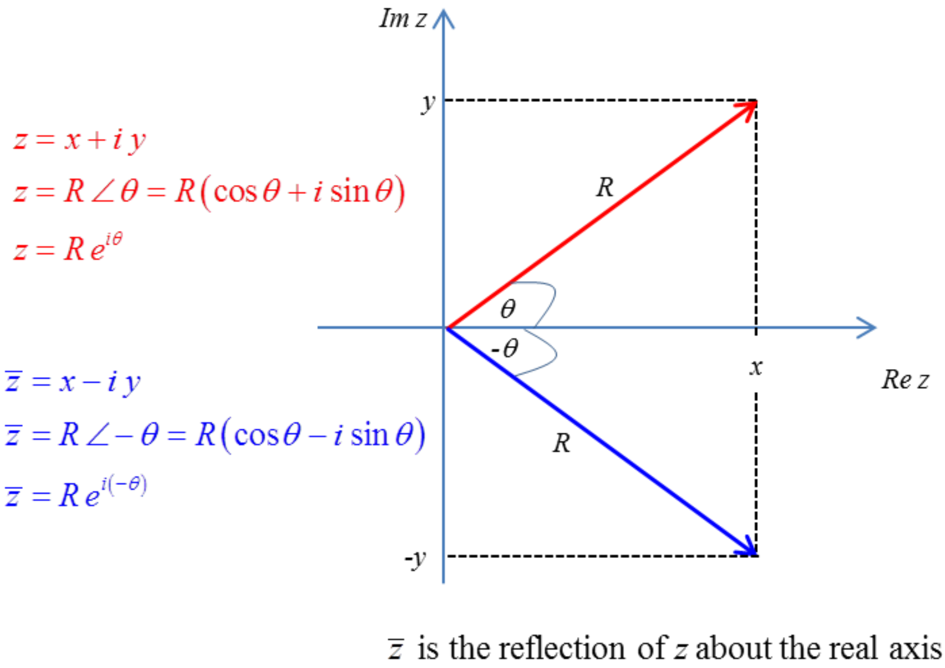

Fig. 2. The complex number z and its complex conjugate $\bar{z}$.

Summary

Rectangular form $z = x + i\ y$

Polar form $z = R(\cos\theta + i\sin\theta)$

Exponential form $z = R\,e^{i\theta}$

Modulus $|z| = \sqrt{z\,\bar{z}} = R = \sqrt{x^2 + y^2}$

Argument

$$\tan\theta = \frac{\operatorname{Im} z}{\operatorname{Re} z} = \frac{y}{x} \qquad \theta = a\tan\left(\frac{y}{x}\right) \equiv \tan^{-1}\left(\frac{y}{x}\right)$$

GEOMETRICAL RELATIONSHIPS

If $z = x + i\,y$ and $x = 0$ and $y = 1$ then $z = i$

$$z = i \qquad \bar{z} = -i \qquad |z| = |i| = 1$$

modulus $\quad |z| = R = |i| = 1$

argument $\quad Arg(z) = \theta = \tan\left(\dfrac{y}{x}\right) = \tan\left(\dfrac{1}{0}\right) \quad = \dfrac{\pi}{2}$

$$z = 1\left(\cos\left(\pi/2\right) + i\,\sin\left(\pi/2\right)\right) = i$$

$$i = e^{i\left(\pi/2\right)}$$

Multiplication of a complex number *z* by a real constant *c*

$$z = x + i\,y = R\left(\cos\theta + i\,\sin\theta\right) = R\,e^{i\theta}$$

$$z_1 = c\,z = c\,x + i\,c\,y = c\,R\left(\cos\theta + i\,\sin\theta\right) = c\,R\,e^{i\theta}$$

modulus of z_1 $\quad |z_1| = c\,R = c\,|z|$

argument of z_1 $\quad \mathrm{Arg}\left(z_1\right) = \mathrm{Arg}\left(z\right) \qquad \theta_1 = \theta$

Multiplication of a complex number by a real constant *c* only changes its modulus by a factor of *c* whist the argument remains the same.

Multiplication of a complex number *z* by *i*

$$z = x + i\ y = R\left(\cos\theta + i\sin\theta\right) = R\,e^{i\,\theta}$$

$$z_1 = i\,z = -y + i\ x$$

$$z_1 = R\left(-\sin\theta + i\cos\theta\right)$$

$$z_1 = R\,i\,e^{i\,\theta} = R\ e^{i\left(\pi/2\right)}e^{i\theta} = R\,e^{i\left(\theta + \pi/2\right)}$$

Modulus $\quad |z_1| = R = |z|$

Argument

$$\arg\left(z_1\right) = \arg\left(z + \frac{\pi}{2}\right) \qquad \tan\theta_1 = \left(\frac{-x}{y}\right) \qquad \tan\theta = \left(\frac{-x}{y}\right) \qquad \theta_1 = \theta + \frac{\pi}{2}$$

Multiplication by i^2

$$z = x + i\ y = R\left(\cos\theta + i\sin\theta\right) = R\,e^{i\,\theta}$$

$$z_2 = i^2\,z = -x - i\ y = -\left(x + i\ y\right) \qquad\qquad i^2 = -1$$

$$z_1 = -\,R\left(\cos\theta + i\sin\theta\right)$$

$$z_1 = R\,i^2\ e^{i\,\theta} = R\ e^{i\left(\pi/2\right)}\,e^{i\left(\pi/2\right)}e^{i\,\theta} = R\,e^{i\,(\theta+\pi)}$$

Modulus $\quad |z_2| = R = |z|$

Argument

$$\arg\left(z_2\right) = \arg\left(z + \pi\right) \qquad \theta_2 = \theta + \pi = -\pi + \theta$$

Multiplication by i^3

$$z = x + i\ y = R(\cos\theta + i\sin\theta) = R e^{i\theta}$$

$$z_3 = i^3 z = -i\ x + y = (y - i\ x) \qquad i^3 = -i$$

$$z_3 = R(\sin\theta - i\cos\theta)$$

$$z_3 = R i^3\ e^{i\theta} = R\ e^{i\left(\pi/2\right)} e^{i\left(\pi/2\right)} e^{i\left(\pi/2\right)} e^{i\theta} = R e^{i\left(\theta + \frac{3\pi}{2}\right)}$$

Modulus $\left|z_2\right| = R = \left|z\right|$

Argument

$$\arg(z_3) = \arg\left(z + \frac{3\pi}{2}\right) \qquad \theta_3 = \theta + \frac{3\pi}{2} = -\frac{\pi}{2} + \theta$$

Multiplication by i^4

$$z = x + i\ y = R(\cos\theta + i\sin\theta) = R e^{i\theta}$$

$$z_4 = i^4 z = x + i\ y = R(\cos\theta + i\sin\theta) = R e^{i\theta} \qquad i^4 = 1$$

$$z_4 = R i^4\ e^{i\theta} = R\ e^{i\left(\pi/2\right)} e^{i\left(\pi/2\right)} e^{i\left(\pi/2\right)} e^{i\left(\pi/2\right)} e^{i\theta} = R e^{i(\theta + 2\pi)}$$

Modulus $\left|z_2\right| = R = \left|z\right|$

Argument

$$\arg(z_4) = \arg(z + 2\pi) = \arg(z) \qquad \theta_4 = \theta + 2\pi = \theta$$

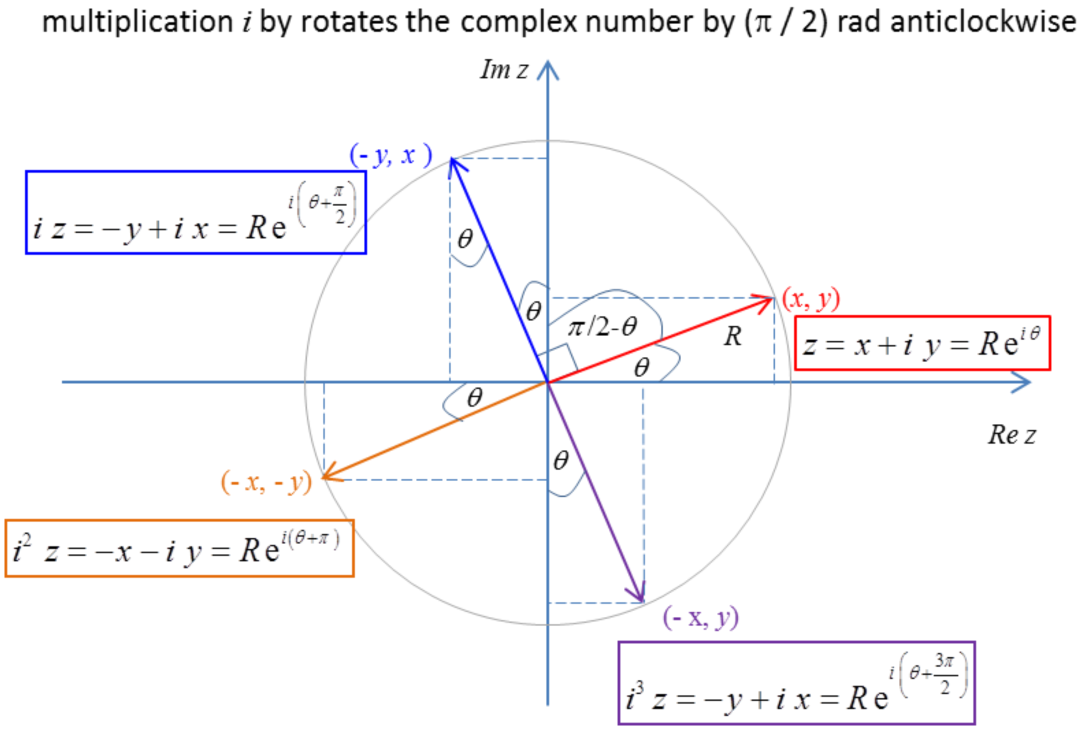

Fig. 3. Multiplication of a complex number by i.