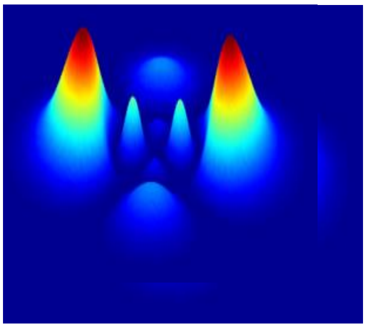

ADVANCED HIGH SCHOOL MATHEMATICS

VOLUMES

EXERCISES

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1

The equation of a circle is given by

$$x^2 + y^2 - 5y + 4 = 0$$

What is the centre of the circle and its radius?

Solution

The given equation of the circle is

$$x^2 + y^2 - 5y + 4 = 0$$

The general form of the equation of a circle with centre (x_C, y_C) and radius a is

$$(x - x_C)^2 + (y - y_C)^2 = a^2$$

We can find (x_C, y_C) and a by comparing the two equations

$$x_C = 0 \quad x^2 + y^2 - 2y_C + y_C{}^2 - a^2 = 0$$

$$y_C = 5/2 \quad y_C{}^2 - a^2 = (5/2)^2 - a^2 = 4$$

$$a = 3/2$$

The centre of the circle is located at (0, 5/2) and the radius is a = 3/2.

2

Find the volume of the solid of revolution generated by the rotation of the curve

$$x^2 + y^2 - 5y + 4 = 0$$

about the X axis.

Solution

The equation of the circle can be written as

$$x^2 + \left(y - 5/2\right)^2 = (3/2)^2$$

The circle can be consider to the constructed from two single-valued curves

$$y_1 = 5/2 + \left[(3/2)^2 - x^2\right]^{1/2}$$

$$y_2 = 5/2 - \left[(3/2)^2 - x^2\right]^{1/2}$$

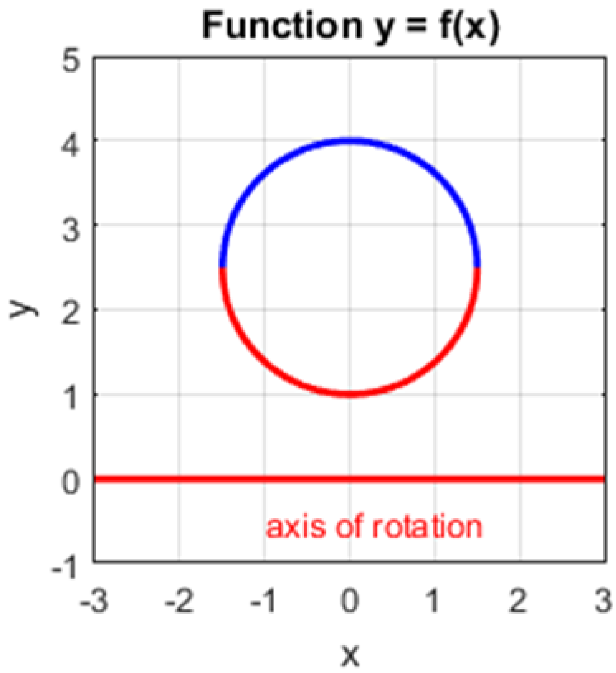

Fig. 1. The axis of rotation is the X axis. The circle has centre (0, 5/2) and radius a = 3/2. The circle can be consider the summation of the two curves y_1 and y_2.

We can use the disk method to find the volume of the solid of revolution by rotating a single valued function $y = f(x)$ about the X axis using the disk method

$$V = \pi \int_{x_B}^{x_A} y^2 \, dx$$

Therefore, the volume V of the solid generated by the rotation of the circle about the X axis is

$$V = \pi \int_{x_B}^{x_A} \left(y_1^{\,2} - y_2^{\,2} \right) dx$$

where x_A = 3/2 and x_B = -3/2 and from the symmetry of the problem, the integral becomes

$$V = 2\pi\int_0^{3/2}\left(y_1^{\ 2} - y_2^{\ 2}\right)dx$$

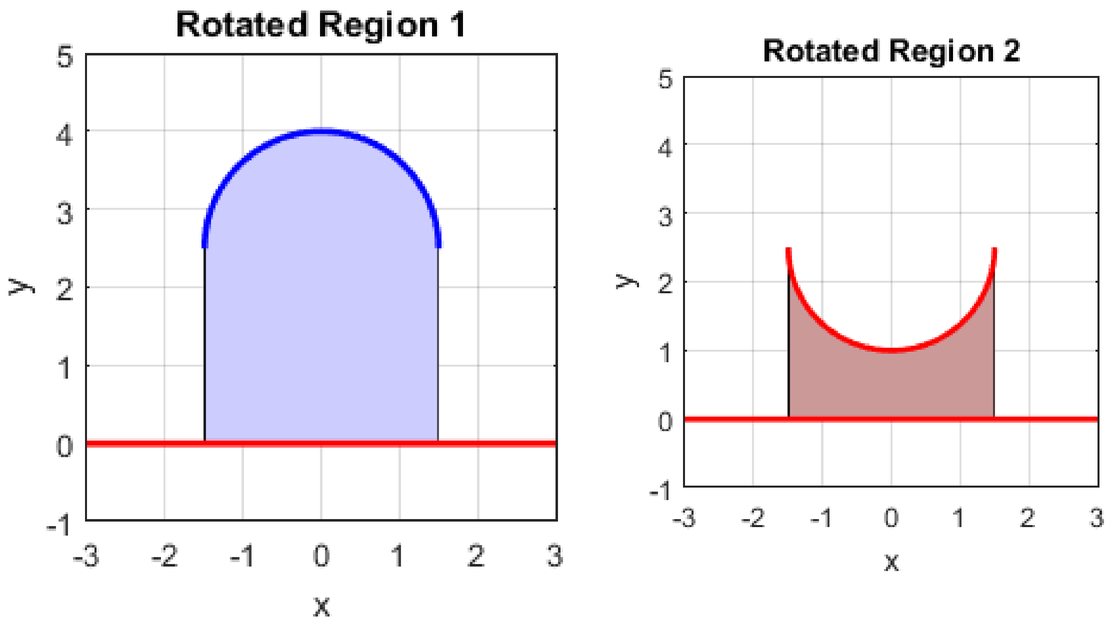

Fig. 2. The volume of the solid of revolution generated is equal to the difference in volumes of the regions formed by the functions y_1 and y_2 when they are rotated about the X axis.

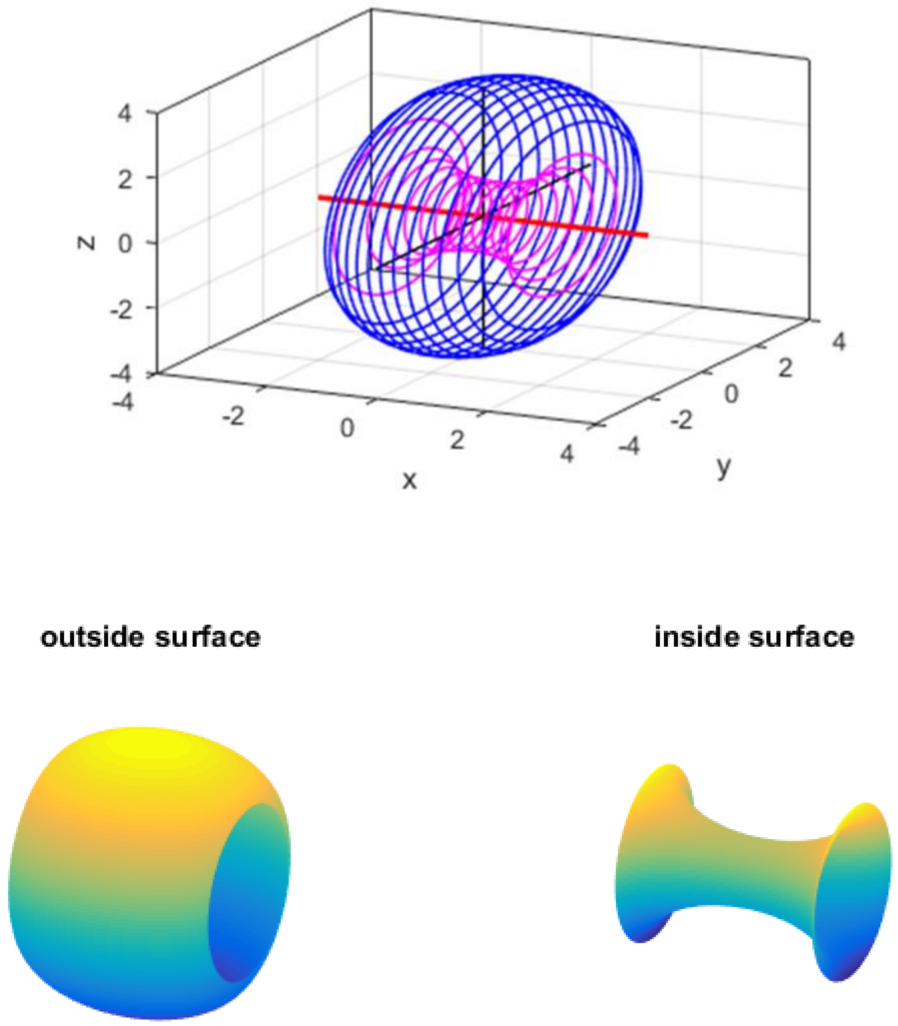

Fig. 3. [3D] plots of the outside and inside surfaces of the solid of revolution.

$$V = 2\pi \int_0^{3/2} \left(y_1^{\,2} - y_2^{\,2} \right) dx$$

$$\begin{aligned} y_1^{\,2} - y_2^{\,2} = {} & (5/2)^2 + 5\left[(3/2)^2 - x^2 \right]^{1/2} + \left[(3/2)^2 - x^2 \right] \\ & - (5/2)^2 + 5\left[(3/2)^2 - x^2 \right]^{1/2} - \left[(3/2)^2 - x^2 \right] \end{aligned}$$

$$y_1^{\,2} - y_2^{\,2} = 10\left[(3/2)^2 - x^2 \right]^{1/2}$$

$$V = 20\pi \int_{x_B}^{x_A} \left[(3/2)^2 - x^2 \right]^{1/2} dx$$

Make the substitution

$$x = (3/2)\sin\theta \quad dx = (3/2)\cos\theta\, d\theta \quad x_A = 3/2 \to \theta_A = \pi/2 \qquad x_B = 0 \to \theta_B = 0$$

$$V = 20\pi \int_0^{\pi/2} (9/4)\left(1-\sin^2\theta\right)^{1/2} \cos\theta\, dx$$
$$V = 45\pi \int_0^{\pi/2} \cos^2\theta\, dx$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$$
$$\cos^2\theta = \tfrac{1}{2}\left(1+\cos(2\theta)\right)$$

$$V = \frac{45\pi}{2}\int_0^{\pi/2}\left(1+\cos(2\theta)\right)dx$$
$$V = \left(\frac{45\pi}{2}\right)\left[\theta + \tfrac{1}{2}\sin(2\theta)\right]_0^{\pi/2}$$

$$V = \frac{45\pi^2}{4}$$

The figures were created using the scientific programming software package MATLAB. The mscript for the figures is **math_vol_07.m** which can be downloaded from

https://drive.google.com/drive/u/3/folders/1j09aAhfrVYpiMavajrgSvUMc89ksF9Jb