



ADVANCED HIGH SCHOOL MATHEMATICS

POLYNOMIALS

EXERCISES

Ian Cooper email: matlabvisualphysics@gmail.com

To find a Question or Answer use the find function: *control f*

Question 4 **Q4**

Answer 15 **A15**

QUESTIONS

Q1

Find the numbers A, B and C such that

$$\frac{2x^2 - 16x - 17}{(x^2 + 5)(x - 6)} = \frac{Ax + B}{x^2 + 5} + \frac{C}{x - 6}$$

Q2

The roots of the quadratic equation $a_1x^2 + b_1x + c_1 = 0$ are $(\alpha, k\alpha)$ and the roots of the equation $a_2x^2 + b_2x + c_2 = 0$ are $(\beta, k\beta)$. Show that

$$a_1 b_2^2 c_1 = a_2 b_1^2 c_2$$

Q3

Show that the necessary and sufficient conditions that the roots of the equation $ax^2 + bx + c = 0$ are real and greater than 1 are

$$b^2 = 4ac > 0 \quad b/a < -2 \quad (b+c)/a > -1$$

Q4

If a, b and c are real constants and $c \neq 0$, show that the roots of the quadratic equation are real and unequal

$$(x-a)(x-b) = c^2$$

If α and β are the roots of this equation, find the equation whose roots are α/β and β/α .

Q5

Find the values of k for which the quadratic equation is a perfect square

$$(x+1)(x+4) + k(x-1)(x-4)$$

Q6

(α, β, γ) are the roots of the cubic equation

$$x^3 + bx^2 + cx + d = 0$$

and

$$\alpha^2 + \beta^2 + \gamma^2 = 14 \quad \alpha^3 + \beta^3 + \gamma^3 = 20 \quad \alpha^4 + \beta^4 + \gamma^4 = 98$$

Determine all the possible values of a , b , and c .

Find a set of possible integer values of the roots (α, β, γ) .

ANSWERS

A1

$$\frac{2x^2 - 16x - 17}{(x^2 + 5)(x - 6)} = \frac{Ax + B}{x^2 + 5} + \frac{C}{x - 6}$$

$$N = (Ax + B)(x - 6) + C(x^2 + 5)$$

$$N = (A + C)x^2 + (-6A + B)x + (-6B + 5C)$$

$$N = 2x^2 - 16x - 17$$

$$A + C = 2 \quad -6A + B = -16 \quad -6B + 5C = -17$$

$$C = 2 - A \quad B + 6C = -4 \quad 6B + 36C = -24$$

$$41C = -41$$

$$A = 3 \quad B = 2 \quad C = -1$$

$$\frac{2x^2 - 16x - 17}{(x^2 + 5)(x - 6)} = \frac{3x + 2}{x^2 + 5} - \frac{1}{x - 6}$$

A2

$$a_1x^2 + b_1x + c_1 = 0$$

$$a_2x^2 + b_2x + c_2 = 0$$

$$\alpha + k\alpha = \alpha(1+k) = -b_1/a_1 \quad k\alpha^2 = c_1/a_1$$

$$\alpha^2 = \frac{b_1^2}{a_1^2(1+k)^2} \quad k\alpha^2 = \frac{kb_1^2}{a_1^2(1+k)^2} = \frac{c_1}{a_1} \quad \frac{k}{(1+k)^2} = \frac{a_1c_1}{b_1^2} = \frac{a_2c_2}{b_2^2}$$

$$a_1b_2^2c_1 = a_2b_1^2c_2 \quad \text{QED}$$

A3

The roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \alpha > \beta > 1$$

For α and β to be real then $b^2 - 4ac > 0$ otherwise the roots will have a non-zero imaginary part.

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} > 1$$

For $\beta > 1$ $\sqrt{b^2 - 4ac} < -(2a + b)$
 $b^2 - 4ac < 4a^2 + 2ab + b^2$
 $\frac{b+c}{a} > -1$

$\alpha > \beta > 1$ therefore $\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} > 2 \quad b/a < -2$

For equal roots $\alpha = \beta = -b/2a > 1 \quad b/a < -2$

A4

$$(x-a)(x-b) = c^2$$
$$x^2 - (a+b)x + ab - c^2 = 0$$

Solve the quadratic equation

$$x = \frac{(a+b) \pm \sqrt{(a+b)^2 - 4(ab - c^2)}}{2}$$

If the roots are unequal, then $\sqrt{(a+b)^2 - 4(ab - c^2)} > 0$

$$\sqrt{(a+b)^2 - 4(ab - c^2)} > 0$$

$$(a+b)^2 - 4(ab - c^2) > 0$$

$$a^2 + b^2 + 2ab - 4ab + 4c^2 > 0$$

$$(a-b)^2 + 4c^2 > 0 \quad (a-b)^2 > 0 \quad 4c^2 > 0 \quad \Rightarrow \quad \text{roots are real and unequal}$$

The equation whose roots are α / β and β / α is

$$(x - \alpha / \beta)(x - \beta / \alpha) = 0$$

$$(x - \alpha / \beta)(x - \beta / \alpha) = 0$$

$$x^2 - (\alpha / \beta + \beta / \alpha)x + 1 = 0$$

$$x^2 - \left(\frac{\alpha^2 + \beta^2}{\alpha\beta} \right)x + 1 = 0$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\alpha + \beta = a + b \quad \alpha\beta = ab - c^2$$

$$x^2 - \left(\frac{(a + b)^2 - 2(ab - c^2)}{ab - c^2} \right)x + 1 = 0$$

$$(a + b)^2 - 2(ab - c^2) = a^2 + b^2 + 2c^2$$

$$(ab - c^2)x^2 - (a^2 + b^2 + 2c^2)x + (ab - c^2) = 1$$

A5

$$f(x) = (x+1)(x+4) + k(x-1)(x-4)$$

$$f(x) = (1+k)x^2 + 5(1-k)x + 4(1+k)$$

A perfect square has equal roots $\alpha = \beta$

$$\alpha + \beta = 2\alpha = \frac{-5(1-k)}{(1+k)} \quad \alpha\beta = \alpha^2 = \frac{4(1+k)}{(1+k)} = 4 \quad \alpha = \pm 2$$

$$k = \frac{-(2\alpha + 5)}{2\alpha - 5}$$

$$\alpha = 2 \quad k = 9 \quad \alpha = -2 \quad k = 1/9$$

$$k = 9 \quad f(x) = 10(x^2 - 4x + 4) = 10(x-2)^2$$

$$k = 1/9 \quad f(x) = (10/9)x^2 + 5(8/9)x + 4(10/9)$$

$$f(x) = (10/9)(x+2)^2$$

A6

$$x^3 + bx^2 + cx + d = 0$$

$$\alpha^2 + \beta^2 + \gamma^2 = 14 \quad \alpha^3 + \beta^3 + \gamma^3 = 20 \quad \alpha^4 + \beta^4 + \gamma^4 = 98$$

$$(1) \quad \alpha + \beta + \gamma = -b \quad (2) \quad \alpha\beta + \alpha\gamma + \beta\gamma = c \quad (3) \quad \alpha\beta\gamma = -d$$

$$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma) \quad b^2 = 14 + 2c$$

$$(4) \quad c = \frac{1}{2}(b^2 - 14)$$

substitute the roots into the cubic equation and then add the 3 equations and use eq(4)

$$\alpha^3 + b\alpha^2 + c\alpha + d = 0$$

$$\beta^3 + b\beta^2 + c\beta + d = 0$$

$$\gamma^3 + b\gamma^2 + c\gamma + d = 0$$

$$(\alpha^3 + \beta^3 + \gamma^3) + b(\alpha^2 + \beta^2 + \gamma^2) + c(\alpha + \beta + \gamma) + 3d = 0$$

$$20 + 14b - \frac{b}{2}(b^2 - 14) + 3d = 0$$

$$(5) \quad b^3 - 42b - 40 - 6d = 0$$

$$x^3 + bx^2 + cx + d = 0 \Rightarrow x^4 + bx^3 + cx^2 + dx = 0$$

$$\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha = 0$$

$$\beta^4 + b\beta^3 + c\beta^2 + d\beta = 0$$

$$\gamma^4 + b\gamma^3 + c\gamma^2 + d\gamma = 0$$

Add the 3 equations and use $\alpha^2 + \beta^2 + \gamma^2 = 14$ $\alpha^3 + \beta^3 + \gamma^3 = 20$ $\alpha^4 + \beta^4 + \gamma^4 = 98$

$$98 + 20b + 14c - bd = 0 \quad \text{replace } c \text{ using eq(4)}$$

$$7b^2 + 20b - bd = 0 \Rightarrow b = 0 \quad \text{is a solution and if } b \neq 0$$

$$(6) \quad d = 7b + 20$$

substitute (6) into (5)

$$b^3 - 84b - 160 = 0 \quad \text{let the roots be } (b_1, b_2, b_3)$$

$$b_1 + b_2 + b_3 = 0 \quad b_1 b_2 b_3 = 160 \Rightarrow (b_1, b_2, b_3) = (10, -8, -2)$$

All possible values of b are $(-8, -2, 0, 10)$

$$\text{From eq(6) } b \neq 0 \quad b = -8 \Rightarrow d = -36 \quad b = -2 \Rightarrow d = 6 \quad b = 10 \Rightarrow d = 90$$

$$\text{From eq(5) } b = 0 \Rightarrow d = -20/3$$

$$\text{From eq(4) } b = -8 \Rightarrow c = 25 \quad b = -2 \Rightarrow c = -5 \quad b = 0 \Rightarrow c = -7 \quad b = 10 \Rightarrow c = 43$$

Consider the set of values $b = -2$ $c = -5$ $d = 6$

$$x^3 + bx^2 + cx + d = 0$$

$$x^3 - 2x^2 - 5x + 6 = 0 \quad \text{set of integer values for the roots are } (1, -2, 3)$$

$$(x - 1)(x + 2)(x - 3) = 0$$