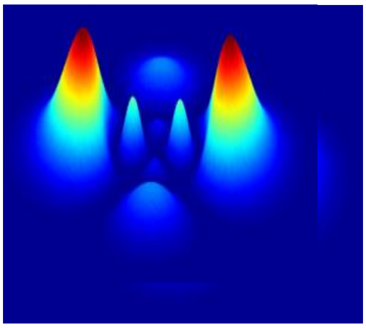

ADVANCED HIGH SCHOOL MATHEMATICS

MECHANICS NUMBERS EXERCISES

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To find a Question or Answer use the find function: *control f*

Question 4   Q4

Answer 15   A15

QUESTIONS

Q1

Consider the Earth as a sphere of radius R_E. At a radial displacement r with respect to the centre of the Earth where $r >> R_E$, the acceleration due to gravity a is directed to the centre of the Earth and inversely proportional to the square of the displacement r. Derive the following results for an object projected vertically with an initial speed v_0 from the surface of the Earth. The magnitude of the acceleration due to gravity at the Earth's surface is g.

The minimum speed for the object to escape from the gravitational field of the Earth and never return is called the **escape speed** v_{esc} where

$$v_{esc} = \sqrt{2\,g\,R_E}$$

The time t_R for the object to reach a distance R_E above the Earth's surface is

$$t_R = \frac{1}{3}\left(\sqrt{\frac{R_E}{g}}\right)\left(4 - \sqrt{2}\right)$$

Q2

Two stones are thrown simultaneously from the same point in the same upward direction at the angle α with respect to the horizontal. The initial velocities of the two stones are u_1 and u_2 where $u_1 < u_2$.The slower stone hits the ground at a point P on the same level as the point of projection. At that instant the faster stone just clears a wall of height h above the level of projection and its downward path makes an angle β with the horizontal.

(a) Show that the line joining them has an inclination with respect to the horizontal that is independent of time when the stones are still both in flight.

(b) What is the horizontal displacement from the point P to the foot of the wall in terms of α and h?

(c) Show that $u_2\left(\tan\alpha + \tan\beta\right) = 2\,u_1\tan\alpha$

(d) If $\alpha = 2\beta$ show that $u_1 < \frac{3}{4}u_2$

Q3

Prove that the range R of a projectile fired upward at an angle θ with initial velocity u is

$$R = \frac{u^2 \sin(2\theta)}{g}$$

where g is the acceleration due to gravity.

A garden sprinkler sprays water about its vertical axis at a constant speed of u. The direction of the spray varies between angles of 15° to 60° with respect to the horizontal.

Prove that from a fixed position O on level ground, the sprinkler will wet the surface of an annular region with centre O with an minimum radius of $u^2/2g$ and a maximum radius u^2/g .

Show that by locating the sprinkler appropriately relative to a rectangular garden bed of dimensions 6 m x 3 m, the entire bed may be water provided that

$$\frac{u^2}{2g} \geq 1 + \sqrt{7}$$

Q4

Consider an object of mass m initially moving with a velocity v_0. It then encounters a resistive force of the form $F_R = -\beta v$ and directed in the opposite direction to the motion. Show that the velocity v and displacement x of the object as functions of time t are $v = v_0 \, e^{-\left(\beta/m\right)t}$

$$x = \left(\frac{m v_0}{\beta}\right)\left(1 - e^{(-\beta/m)t}\right)$$

What are the values of the velocity and displacement as $t \to \infty$?

Plot velocity and displacement time graphs for the parameters

$v_0 = 10$ $m = 2$ $\beta = 5$

$v_0 = 5$ $m = 2$ $\beta = 5$

$v_0 = 10$ $m = 4$ $\beta = 5$

$v_0 = 10$ $m = 2$ $\beta = 10$

Comment on the physical significant of these changes in parameters.

Q5

A ball bearing was released from rest and dropped through a viscous liquid. The resistive force acting on the ball had magnitude $k\,v$ where k is a constant depending on the radius of the ball and the viscosity of the liquid and v is the velocity of the ball.

Find the following:

The terminal velocity v_T of the ball

t as a function of v

v as a function of t

The time it takes for the ball to reach a speed equal to half its terminal speed

Q6

Consider an object of mass m falling due to gravity. The object was released with an initially velocity v_0. The resistive force due to the medium the object falls through is of the form $F_R = -\beta v$ and directed in the opposite direction to the motion. Derive the following results

$$v_T = \frac{m\,g}{\beta}$$

$$a = \left(\frac{\beta}{m}\right)\left(v_T - v_0\right)e^{(-\beta/m)\,t}$$

$$v = v_T + \left(v_0 - v_T\right)e^{(-\beta/m)\,t}$$

$$x = v_T\,t + \left(\frac{m}{\beta}\right)\left(v_T - v_0\right)e^{(-\beta/m)\,t}$$

$$x = \left(\frac{m}{b}\right)\left(\left(v_0 - v\right) + v_T\,\log_e\left(\frac{v_T - v_0}{v_T - v}\right)\right)$$

Comment on the acceleration a, velocity v and displacement x as $t \to \infty$?

Sketch graphs for acceleration a, velocity v and displacement x time graphs for

$v_0 > v_T$ $v_0 = 0$ and $v_0 < v_T$ where v_0 is the initial velocity.

Q7

Consider the vertical motion an object of mass m near the Earth's surface.The object was released with an initially velocity v_0. The resistive force due acting on the object is of the form $F_R = -\alpha v^2$ and directed in the opposite direction to the motion. Derive the following result and comment on the acceleration a, velocity v and displacement x as $t \to \infty$? Down is the positive direction.

Object falling $v > 0$ only $a = g - \frac{\alpha}{m} v^2$

$$v_T = \sqrt{\frac{m\,g}{\alpha}}$$

$$v = v_T \left(\frac{(v_0 + v_T) + (v_0 - v_T)\, e^{-\frac{2g}{v_T} t}}{(v_0 + v_T) - (v_0 - v_T)\, e^{-\frac{2g}{v_T} t}} \right)$$

$$x = \left(\frac{V_T^{\,2}}{2\,g} \right) \log_e \left(\frac{v_T^{\,2} - v_0^{\,2}}{v_T^{\,2} - v^2} \right)$$

Object rising $v < 0$ only $a = g + \frac{\alpha}{m} v^2$

$$v = v_T \,\tan\left[\operatorname{atan}\left(\frac{v_0}{v_T} \right) + \left(\frac{g}{v_T} \right) t \right]$$

$$x = \left(\frac{v_T^{\,2}}{2\,g} \right) \log_e \left(\frac{v_T^{\,2} + v^2}{v_T^{\,2} + v_0^{\,2}} \right)$$

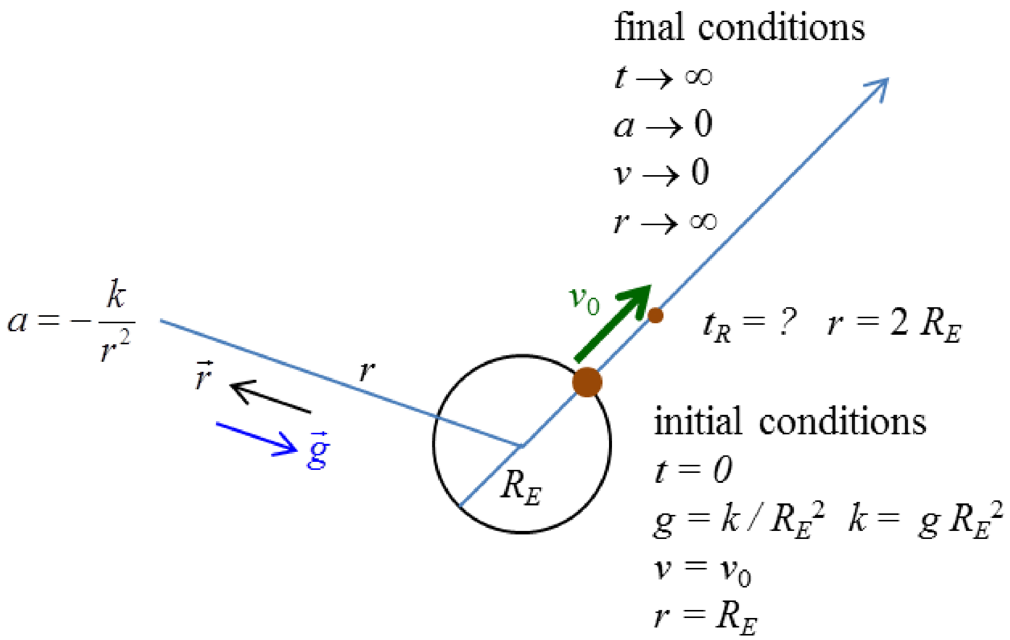
ANSWERS

A1

Step 1: Think about how to approach the problem

Step 2: Draw an annotated diagram of the physical situation

Step 3: What do you know about displacement, velocity and acceleration?



Acceleration a

$$a = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)$$

$$a = \frac{d}{dr}\left(\tfrac{1}{2}v^2\right) = -\frac{k}{r^2}$$ where k = constant of proportionality

Integrating the acceleration a with respect to r where the limits of the integration are determined by the initial conditions ($v = v_0 \quad r = R_E$) and final conditions ($v = 0 \quad r = \infty$)

$$a = \frac{dv}{dr}\left(\tfrac{1}{2}v^2\right) = -\frac{k}{r^2} \quad g = \left|-\frac{k}{R_E{}^2}\right| = \frac{k}{R_E{}^2} \quad k = g\,R_E{}^2$$

$$\int_{v_0}^{0} d\left(\tfrac{1}{2}v^2\right) = \int_{R_E}^{\infty}\left(-\frac{gR_E{}^2}{r^2}\right)dr$$

$$\left[\tfrac{1}{2}v^2\right]_{v_0}^{0} = \left[\frac{gR_E{}^2}{r}\right]_{R_E}^{\infty}$$

$$-\tfrac{1}{2}v_0{}^2 = 0 - \frac{gR_E{}^2}{R_E}$$

$$v_0 = \sqrt{2gR_E}$$

Hence, the escape velocity is $v_{esc} = \sqrt{2gR_E}$ ***QED***

The integration of the acceleration gives the velocity v as a function of r

$$v = \sqrt{2gR_E{}^2}\,\left(r\right)^{-1/2}$$

The velocity v is the time derivative of the displacement r

$$v = \frac{dr}{dt} = \sqrt{2gR_E{}^2}\,\left(r\right)^{-1/2}$$

To find the time interval t_R it takes for the displacement of the object to go from $r = R_E$ to $r = 2\,R_E$ is found by integration of the above equation where the lower limits are ($t = 0 \quad r = R_E$) and the upper limits are ($t = t_R \quad r = 2\,R_E$)

$$dt = \left(\frac{1}{\sqrt{2g}\ R_E} \right) (r)^{1/2}\, dr$$

$$\int_0^{t_R} dt = \int_{R_E}^{2R_E} \left(\frac{1}{\sqrt{2g}\ R_E} \right) (r)^{1/2}\, dr$$

$$t_R = \left(\frac{1}{\sqrt{2g}\ R_E} \right) \left(\frac{2}{3} \right) \left[r^{3/2} \right]_{R_E}^{2R_E}$$

$$t_R = \left(\frac{1}{\sqrt{2g}\ R_E} \right) \left(\frac{2}{3} \right) \left(2\sqrt{2}\ R_E{}^{3/2} - R_E{}^{3/2} \right)$$

$$t_R = \left(\frac{1}{3} \right) \left(\sqrt{\frac{R_E}{g}} \right) \left(\sqrt{2} \right) \left(2\sqrt{2} - 1 \right)$$

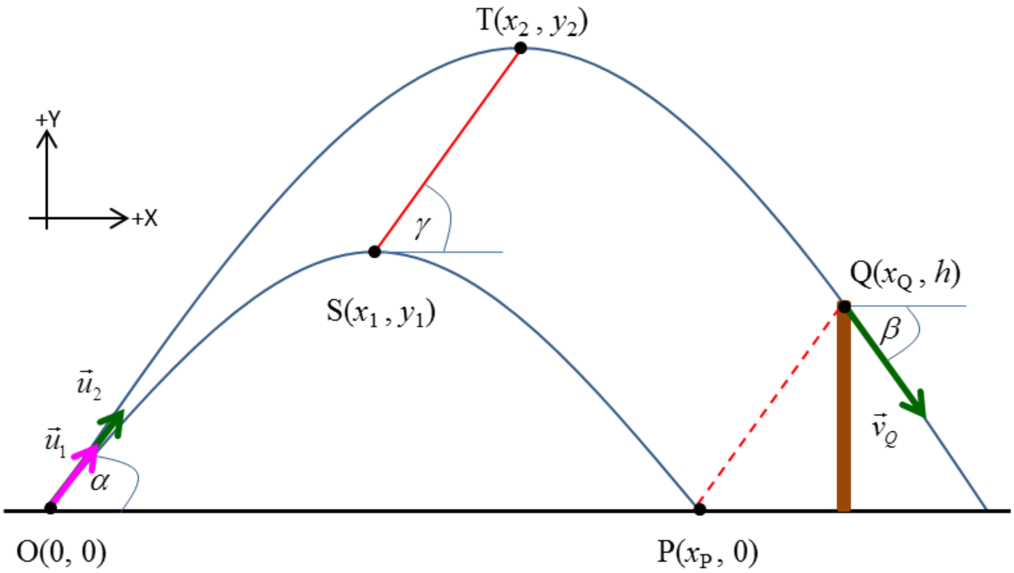
$$t_R = \left(\frac{1}{3} \right) \left(\sqrt{\frac{R_E}{g}} \right) \left(4 - \sqrt{2} \right)$$

A2

Step 1: Think about how to approach the problem

Step 2: Draw an annotated diagram of the physical situation

Step 3: What do you know about projectile motion and motion with a constant acceleration?



The origin is at 0(0, 0).

Stone 1 has initial velocity u_1

$$u_{1x} = u_1 \cos\alpha \quad u_{1y} = u_1 \sin\alpha \quad \tan\alpha = \left(u_{1y} / u_{1y}\right)$$

Stone 2 has initial velocity u_2 $\quad u_2 > u_1$

$$u_{2x} = u_2 \cos\alpha \quad u_{2y} = u_1 \sin\alpha \quad \tan\alpha = \left(u_{2y} / u_{2y}\right)$$

Stone 1 lands at the point P(x_P, 0). At this instance stone 2 is at the point Q(x_Q, h).

The velocity of stone 2 at the point Q is $\vec{v}_Q$

$$v_{Qx} = v_Q \cos\alpha \quad u_{Qy} = v_Q \sin\alpha \quad \tan\beta = \left(v_{Qy} / v_{Qx}\right)$$

(a)

When stone 1 is at the point S(x_1, y_1) then stone 2 is at the point T(x_2, y_2).

The inclination of the line joining the stones 1 and 2 makes an angle γ with the horizontal where

$$\tan\gamma = \frac{y_2 - y_1}{x_2 - x_1}$$

Therefore, we need to find the coordinates of the points S(x_1, y_1) and T(x_2, y_2) at some instance t.

$$a_{1x} = 0 \quad a_{1y} = -g \quad v_{1y} = 0$$

Motion with constant acceleration $\quad v = u + a\,t \quad v^2 = u^2 + 2a\,s \quad s = u\,t + \tfrac{1}{2}a\,t^2$

$$x_1 = u_{1x}\,t = \left(u_1 \cos\alpha\right)t \quad y_1 = u_{1y}\,t + \tfrac{1}{2}\left(-g\right)t^2 = \left(u_1 \sin\alpha\right)t - \tfrac{1}{2}g\,t^2$$

$$x_2 = \left(u_2 \cos\alpha\right)t \quad y_2 = \left(u_2 \sin\alpha\right)t - \tfrac{1}{2}g\,t^2$$

$$y_2 - y_1 = \left(u_2 \sin\alpha\right)t - \tfrac{1}{2}g\,t^2 - \left(\left(u_1 \sin\alpha\right)t - \tfrac{1}{2}g\,t^2\right) = \left(u_2 - u_1\right)\sin\alpha\ t$$

$$x_2 - x_1 = \left(u_2 \cos\alpha\right)t - \left(u_1 \cos\alpha\right)t = \left(u_2 - u_1\right)\cos\alpha\ t$$

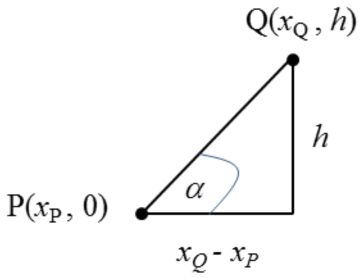
$$\tan\gamma = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\left(u_2 - u_1\right)\sin\alpha\ t}{\left(u_2 - u_1\right)\cos\alpha\ t} = \tan\alpha$$

$$\gamma = \alpha$$

The inclination angle is α hence, the inclination is independent of time.

(b)

The inclination angle of the lining joining the stones is independent of time or position



$$\tan\alpha = \frac{h}{x_Q - x_P}$$

The distance from the landing point of stone 1 at the point P to the foot of the wall is

$$x_Q - x_P = h \tan\alpha$$

(c) Show that $u_2\left(\tan\alpha + \tan\beta\right) = 2u_1 \tan\alpha$

Rearranging this equation gives (A) $\tan\beta = \left(\dfrac{2u_1}{u_2} - 1\right)\tan\alpha$

Using the equations for constant acceleration at the time *t* that stone 1 hits the ground at the point P

Stone 1 (vertical motion)

$$s = ut + \tfrac{1}{2}a t^2 \qquad 0 = u_1 \sin\alpha\, t - \tfrac{1}{2} g t^2 \qquad t = \frac{2u_1 \sin\alpha}{g}$$

Stone 2 $v_{2x} = u_2 \cos\alpha$

$$v = u + a\,t \qquad v_{2y} = u_2 \sin\alpha - g\left(\frac{2\,u_1 \sin\alpha}{g}\right)$$

$$v_{2y} = (u_2 - 2u_1)(\sin\alpha)$$

$$\tan\beta = -\frac{v_{2y}}{v_{2x}} = -\frac{(u_2 - 2u_1)(\sin\alpha)}{u_2 \cos\alpha}$$

$$\tan\beta = \left(2\,\frac{u_1}{u_2} - 1\right)\tan\alpha$$

Which is the result we need to show as given by equation (A).

(d)

$$\beta = \alpha/2$$

$$\tan\beta = \left(2\,\frac{u_1}{u_2} - 1\right)\tan\alpha$$

$$\tan(\alpha/2) = \left(2\,\frac{u_1}{u_2} - 1\right)\tan(\alpha/2 + \alpha/2)$$

$$\tan(\alpha/2) = \left(2\,\frac{u_1}{u_2} - 1\right)\left(\frac{\tan(\alpha/2) + \tan(\alpha/2)}{1 - \tan^2(\alpha/2)}\right)$$

$$\text{let} \quad z = \tan(\alpha/2) \qquad K = \left(2\,\frac{u_1}{u_2} - 1\right)$$

$$z - z^3 = 2\,K\,z \qquad z^2 = 1 - 2K$$

But $z = \tan(\alpha/2)$ must be a real quantity

$$2K < 1$$

$$2\left(2\,\frac{u_1}{u_2} - 1\right) < 1$$

$$4\,\frac{u_1}{u_2} < 3$$

$$u_1 < \tfrac{3}{4}\,u_2$$

A3

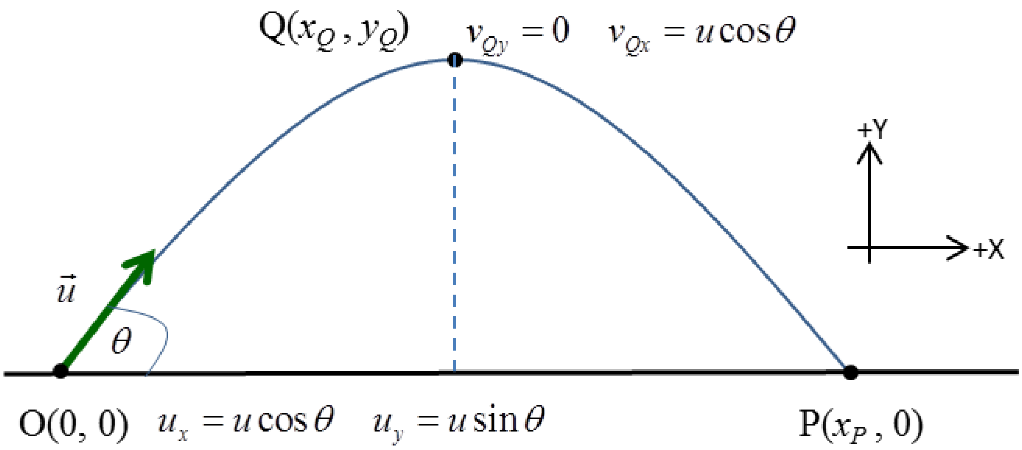
Step 1: Think about how to approach the problem

Step 2: Draw annotated diagrams of the physical situations

Step 3: What do you know about projectile motion and motion with a constant acceleration?

We can consider the motion in the +X direction to be independent of the motion in the Y direction and use the equation for constant acceleration

$$v = u + a\,t \qquad s = u\,t + \tfrac{1}{2}a\,t^2 \qquad v^2 = u^2 + 2\,a\,s$$



First, calculate the time t_Q it takes for the projectile to reach its maximum height at Q.

Horizontal motion: X direction	Vertical motion: Y direction
$v_{Qx} = u\cos\theta$ $x_Q = v_x\, t = u\cos\theta\, t$ $x_Q = \left(u\cos\theta\right)\left(\dfrac{u\sin\theta}{g}\right) = \dfrac{u^2\, \cos\theta\, \sin\theta}{g}$	$a_y = -g \quad v_{Qy} = 0$ $0 = u\sin\theta - g\, t$ $t = \dfrac{u\sin\theta}{g}$

The range of the projectile is $R = x_P = 2x_Q$

$$R = \frac{u^2\, (2)\cos\theta\, \sin\theta}{g} = \frac{u^2\, \sin(2\theta)}{g}$$

The maximum range occurs when $\sin(2\theta) = 1 \quad \theta = 45^o$

$$R_{\max} = \frac{u^2}{g}$$

The minimum range is

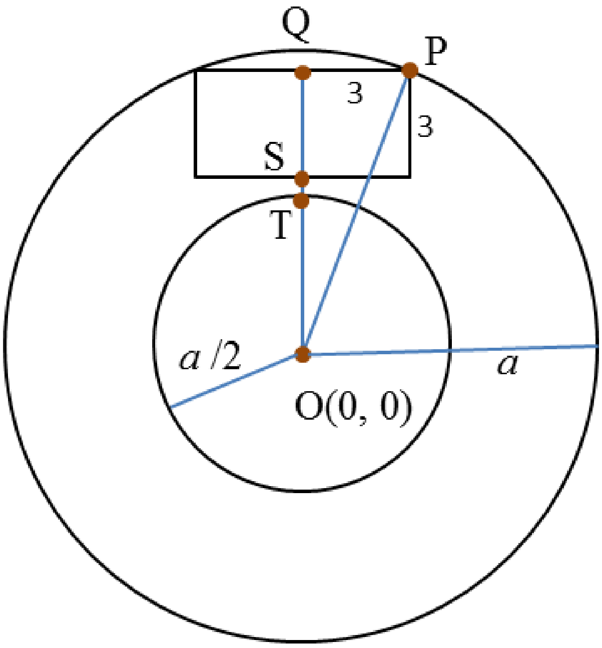
$$R\left(\theta = 15^o\right) = \frac{u^2\, \sin\left(30^o\right)}{g} = \frac{u^2}{2g}$$

$$R\left(\theta = 60^o\right) = \frac{u^2\, \sin\left(120^o\right)}{g} = \frac{0.866\, u^2}{g}$$

$$R_{\min} = \frac{u^2}{2g} \quad \theta = 15^o$$

The rectangular garden bed 6 m x 3 m must fit into the region bounded by the two circles. Let *a* be the radius of the larger circle and *a*/2 be the radius of the smaller circle

$$a = \frac{u^2}{g} \qquad a/2 = \frac{u^2}{2g}$$



Hence, we require that the distance OT to be less than the distance OS.

We need to find the coordinates of the points P, Q, S and T.

Point P lies on the circle $x^2 + y^2 = a^2$

$$x_P = 3 \quad y_P = \sqrt{a^2 - 9}$$

Point Q $\quad x_Q = 0 \quad y_Q = \sqrt{a^2 - 9}$

Point S $\quad x_S = 0 \quad y_S = \sqrt{a^2 - 9} - 3$

Point T $\quad x_T = 0 \quad y_Q = a/2$

We require that OS ≥ OT

$$\sqrt{a^2-9}-3 \ge a/2$$
$$2\sqrt{a^2-9} \ge 6+a$$
$$4\left(a^2-9\right) \ge 36+12a+a^2$$
$$3a^2-12a-72 \ge 0$$
$$a^2-4a-24 \ge 0$$

We need to solve the quadratic equation $a^2-4a-24=0$ to find a

$$a=\left(\tfrac{1}{2}\right)\left(4+\sqrt{16+(4)(1)(24)}\right)=\left(\tfrac{1}{2}\right)\left(4+\sqrt{16+(4)(1)(4)(6)}\right)$$
$$a=\left(\tfrac{1}{2}\right)\left(4+4\sqrt{1+6}\right)=2+2\sqrt{7}$$

Hence

$$a/2=\frac{u^2}{2g} \ge 1+\sqrt{7}$$

A4

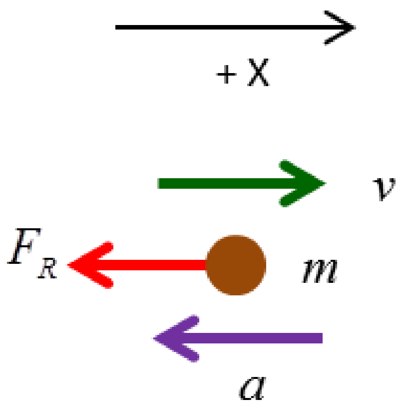
The force acting on the object is the resistive force F_R. In our frame of reference, we will take to the right as the positive direction.

The equation of motion of the object is determined from Newton's Second Law.

$$m\,a = m\,\frac{dv}{dt} = F_R = -\beta\,v$$

where a is the acceleration of the object at any instance.

The initial conditions are $t = 0 \quad v = v_0 \quad x = 0 \quad a = -(\beta / m)\,v_0$



We start with the equation of motion then integrate this equation where the limits of the integration are determined by the initial conditions (t = 0 and $v = v_0$) and final conditions (t and v)

$$\frac{dv}{dt} = -\left(\frac{\beta}{m}\right)v$$

$$\frac{dv}{v} = -\left(\frac{\beta}{m}\right)dt$$

$$\int_{v_0}^{v} \frac{dv}{v} = \int_0^t -\left(\frac{\beta}{m}\right)dt$$

$$\left[\log_e(v)\right]_{v_0}^{v} = -\left(\frac{\beta}{m}\right)t$$

$$\log_e\left(\frac{v}{v_0}\right) = -\left(\frac{\beta}{m}\right)t$$

$$v = v_0\, e^{(-\beta/m)\,t}$$

We can now calculate the displacement x as a function of time t

$$v = \frac{dx}{dt} \quad dx = v\,dt$$

$$v = v_0\, e^{(-\beta/m)\,t}$$

$$\int_0^x dx = \int_{v_0}^{v} v_0\, e^{(-\beta/m)\,t}\, dt$$

$$x = -\left(\frac{m\,v_0}{\beta}\right)\left[e^{(-\beta/m)\,t}\right]_0^t$$

$$x = \left(\frac{m\,v_0}{\beta}\right)\left(1 - e^{(-\beta/m)\,t}\right)$$

The velocity v also can be given as a function of x

$$a = \frac{dv}{dt} = v\frac{dv}{dx} = -\frac{\beta}{m}v \qquad dv = -\frac{\beta}{m}dx$$

$$\int_{v_0}^{v} dv = \left(-\frac{\beta}{m}\right)\int_0^x dx \qquad v = v_0 - \frac{\beta}{m}x$$

The graph of v vs x is a straight line.

When v = 0 the stopping distance is $x_{stopping} = \dfrac{m\,v_0}{\beta}$

We can also derive the result from

$$a = v\frac{dv}{dx} = -\left(\frac{\beta}{m}\right)v$$
$$dv = -\left(\frac{\beta}{m}\right)dx$$
$$\int_{v_0}^{v} dv = -\left(\frac{\beta}{m}\right)\int_0^x dx$$
$$-\left(\frac{\beta}{m}\right)x = v - v_0$$
$$x = \left(\frac{m\,v_0}{\beta}\right)\left(1 - e^{(-\beta/m)\,t}\right)$$

We can now investigate what happens when $t \to \infty$

$$t \to \infty \quad v = v_0\,e^{(-\beta/m)\,t} \to 0$$
$$t \to \infty \quad x = \left(\frac{m\,v_0}{\beta}\right)\left(1 - e^{(-\beta/m)\,t}\right) \to \frac{m\,v_0}{\beta}$$

The object keeps moving till v = 0, which happens only in the limit $t \to \infty$. Then the object stop at the position $x = \dfrac{m\,v_0}{\beta}$.

We can define a **time constant** τ

$$\tau = \frac{\beta}{m}$$

The velocity and displacement can be expressed as

$$v = v_0\, e^{-t/\tau}$$

$$x = \left(\frac{m\, v_0}{\beta}\right)\left(1 - e^{-t/\tau}\right)$$

After a time of about 5τ, the particle will stop when the speed of the particle becomes zero

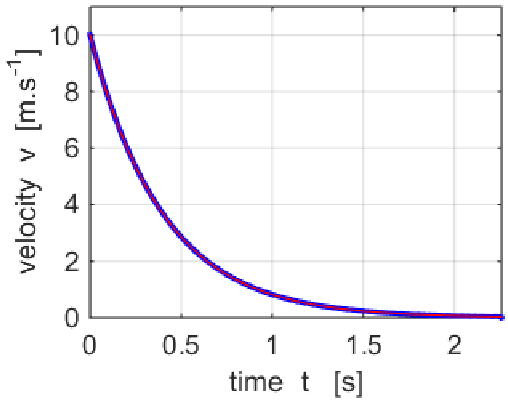
$$x_{final} = \left(\frac{m\, v_0}{\beta}\right) \qquad \text{stopping time} \sim 5\ \tau$$

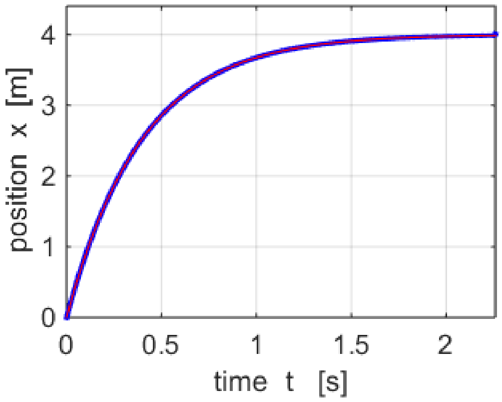
The stopping time is independent of the initial velocity but the greater the initial velocity the greater the stopping distance. The larger the constant β, the shorter the stopping distance and quicker it stops and the larger the mass, the greater the stopping distance and it takes a longer time to stop the object.

The following graphs show the velocity and displacement as functions of time for varying values of v_0, m and β.

$v_0 = 10$ $m = 2.0$ $\beta =$ $\tau = 0.4$ $5\tau = 2.0$ $x_{final} = 4.0$

5.0

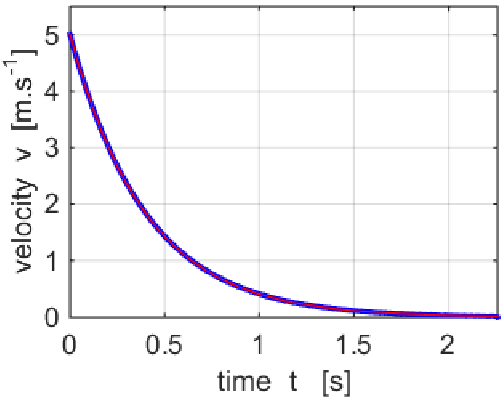


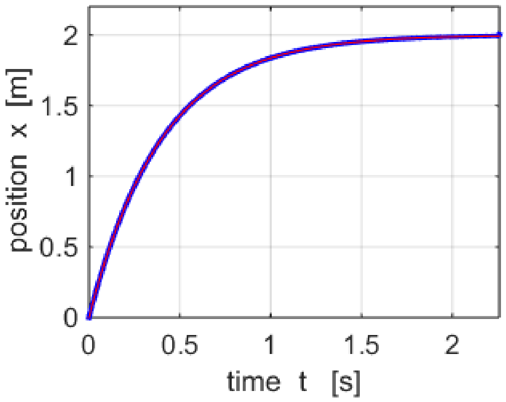


$v_0 = 5.0$ $m = 2.0$ $\beta =$ $\tau = 0.4$ $5\tau = 2.0$ $x_{final} =$

5.0

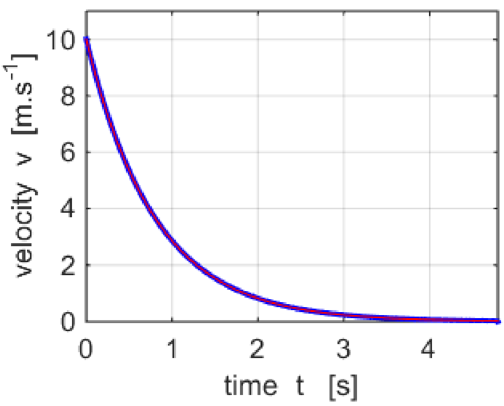
2.0

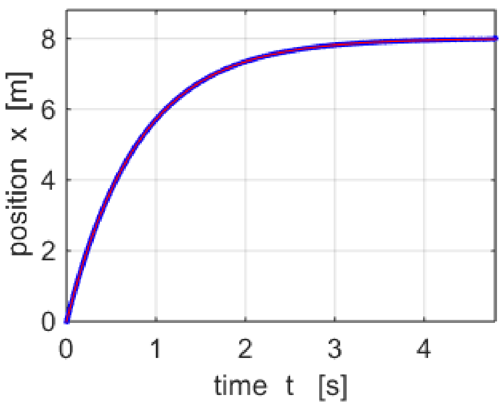




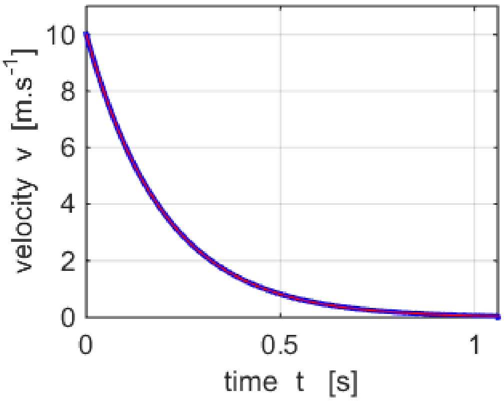
$v_0 = 10$ $m = 4.0$ $\beta =$ $\tau = 0.8$ $5\tau = 4.0$ $x_{final} = 8.0$

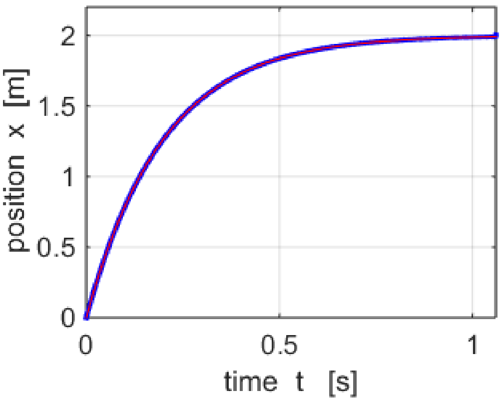
5.0



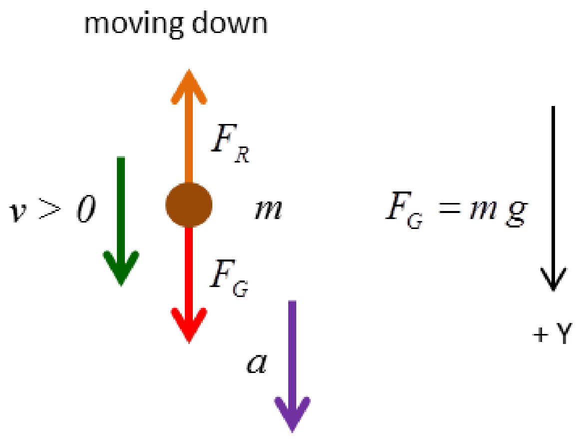


$v_0 = 10$ $m = 2.0$ $\beta = 10$ $\tau = 0.2$ $5\tau = 1.0$ $x_{final} = 2.0$





A5



The forces acting on the ball as it falls through the liquid are the gravitational force F_G and the resistive force F_R. In our frame of reference, we will take down as the positive direction.

The equation of motion of the ball is determined from Newton's Second Law.

$$m\,a = m\,g - k\,v$$

where a is the acceleration of the ball at any instance and g is the acceleration due to gravity.

The initial conditions are $\quad t = 0 \quad a = g \quad v = 0$

As the ball falls, value of v increases until it reaches its terminal speed $v = v_T$ when the acceleration becomes zero

$$a = 0 \quad v = v_T = \text{constant}$$
$$0 = m\,g - k\,v_T$$
$$v_T = \frac{m\,g}{k}$$

We start with the equation of motion

$$m\frac{dv}{dt} = m\,g - k\,v$$
$$dt = \frac{m\,dv}{m\,g - k\,v} = \frac{dv}{g - (k/m)v}$$

then integrate this equation where the limits of the integration are determined by the initial conditions (t = 0 and v = 0) and final conditions (t and v)

$$\int_0^t dt = \int_0^v \frac{dv}{g - (k/m)v}$$
$$t = \int_0^v \frac{dv}{g - (k/m)v}$$

The integration can be done by making the substitution

$$u = g - (k/m)v$$
$$du = -(k/m)dv \quad dv = -(m/k)du$$

and the now limits of the integration are

$$v = 0 \rightarrow \quad u = g \quad v \rightarrow \quad u = g - (k/m)v$$

$$t = (-m/k)\int_g^{g-(k/m)v} \frac{du}{u}$$
$$t = (-m/k)\left[\log_e(u)\right]_g^{g-(k/m)v}$$
$$t = (-m/k)\left[\log_e\left(g - (k/m)v\right) - \log_e(g)\right]$$
$$t = (-m/k)\log_e\left(1 - (k/mg)v\right)$$

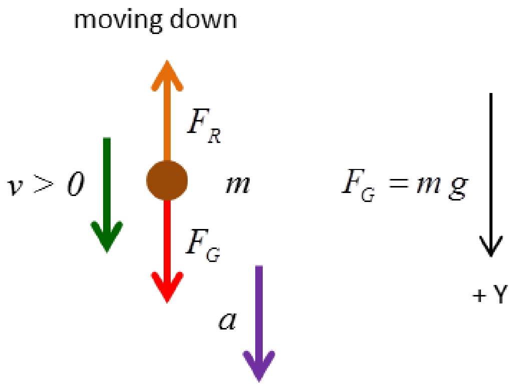
The equation of the velocity as a function of time t is

$$e^{(-k/m)t} = 1 - (k/m\,g)v$$
$$v = \left(\frac{m\,g}{k}\right)\left(1 - e^{(-k/m)t}\right)$$
$$v = v_T\left(1 - e^{(-k/m)t}\right)$$

The time to reach half the terminal velocity is

$$
\begin{aligned}
&v = v_T / 2 \\
&v_T / 2 = v_T \left(1 - e^{(-k/m)t}\right) \\
&e^{(-k/m)t} = 1/2 \\
&\left(-k/m\right) t = \log_e(1/2) \\
&t = \left(\frac{m}{k}\right) \log_e(2)
\end{aligned}
$$

A6



The forces acting on the object are the gravitational force F_G (weight) and the resistive force F_R. In our frame of reference, we will take down as the positive direction.

The equation of motion of the object is determined from Newton's Second Law.

$$m\,a = m\,\frac{dv}{dt} = F_G - F_R = m\,g - \beta\,v$$

where a is the acceleration of the object at any instance.

The initial conditions are $t = 0 \quad v = v_0 \quad x = 0 \quad a = g - \left(\frac{\beta}{m}\right) v_0$

When $a = 0$, the velocity is constant $v = v_T$ where v_T is the terminal velocity

$$0 = m\,g - \beta\,v_T$$

$$v_T = \frac{m\,g}{\beta}$$

We start with the equation of motion then integrate this equation where the limits of the integration are determined by the initial conditions (t = 0 and $v = v_0$) and final conditions (t and v)

$$a = \frac{dv}{dt} = g - \left(\frac{\beta}{m}\right)v = -\left(\frac{\beta}{m}\right)\left(v - \frac{m\,g}{\beta}\right)$$

$$u = v - \frac{m\,g}{\beta} \qquad du = dv \qquad -\left(\frac{\beta}{m}\right)dt = \frac{du}{u}$$

$$-\left(\frac{\beta}{m}\right)\int_0^t dt = \int_{u0}^{u} \frac{du}{u} \qquad -\left(\frac{\beta}{m}\right)t = \left[\log_e\left(u\right)\right]_{v_0 - \frac{m\,g}{\beta}}^{v - \frac{m\,g}{\beta}} = \log_e\left(\frac{v - \frac{m\,g}{\beta}}{v_0 - \frac{m\,g}{\beta}}\right)$$

$$\left(\frac{v - \frac{m\,g}{\beta}}{v_0 - \frac{m\,g}{\beta}}\right) = e^{(-\beta/m)\,t}$$

$$v = \frac{m\,g}{\beta} + \left(v_0 - \frac{m\,g}{\beta}\right)e^{(-\beta/m)\,t} \qquad v_T = \frac{m\,g}{\beta}$$

$$v = v_T + \left(v_0 - v_T\right)e^{(-\beta/m)\,t}$$

$$v_0 = 0 \quad \Rightarrow \quad v = v_T\left(1 - e^{(-\beta/m)t}\right)$$

$$v_0 = v_T \Rightarrow \quad v = v_T$$

$$v_0 < v_T \Rightarrow \quad v \text{ increases to } v_T$$

$$v_0 > v_T \Rightarrow \quad v \text{ decreases to } v_T$$

In every case, the velocity v tends towards the limiting value vT.

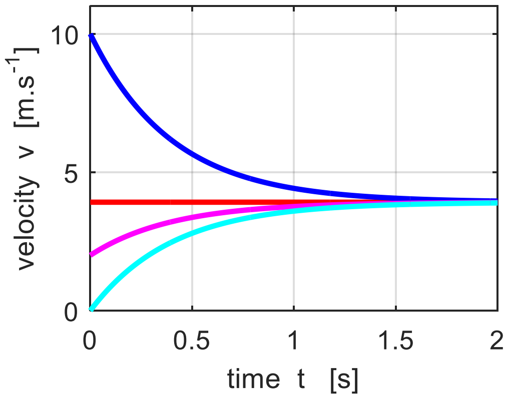
Plots of the velocity v as a function of time t

m = 2.00 kg

β = 5.00 kg.s^{-1}

g = 9.80 m.s^{-2}

v_T = 3.92 m.s^{-1}



Initial values for velocity v_0 [m.s^{-1}]

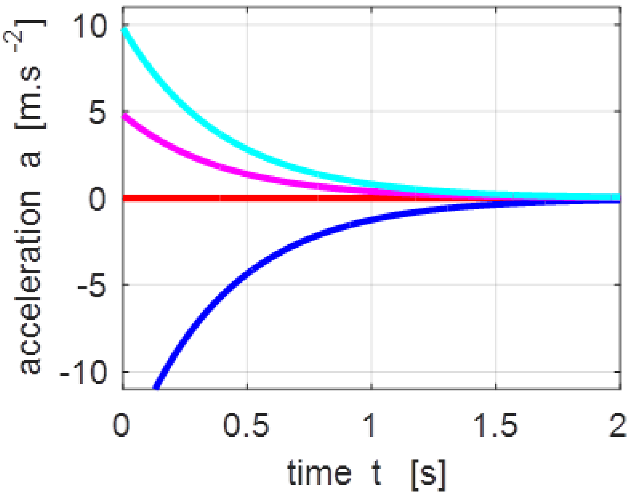
blue: 10 red: v_T magenta: 2 cyan: 0

The acceleration a as a function of time t is

$$v = v_T + (v_0 - v_T)e^{(-\beta/m)t}$$
$$a = \frac{dv}{dt} = \frac{d}{dt}\left(v_T + (v_0 - v_T)e^{(-\beta/m)t}\right)$$
$$a = (v_0 - v_T)\left(\frac{-\beta}{m}\right)e^{(-\beta/m)t}$$
$$a = (v_T - v_0)\left(\frac{\beta}{m}\right)e^{(-\beta/m)t}$$

$$v_0 = 0 \quad \Rightarrow \quad a = \left(\frac{\beta\, v_T}{m}\right)e^{(-\beta/m)t}$$
$$\Rightarrow \quad a = g\; e^{(-\beta/m)t}$$
$$v_0 = v_T \Rightarrow \quad a = 0$$
$$v_0 < v_T \Rightarrow \quad a > 0 \quad \text{and decreases to } 0$$
$$v_0 > v_T \Rightarrow \quad a < 0 \quad \text{and } a \text{ increases to } 0$$
$$t \to \infty \Rightarrow \quad a \to 0$$

Initial values for velocity v_0 [m.s^{-1}]



blue: 10 red: v_T magenta: 2 cyan: 0

We can now calculate the displacement x as a function of velocity t

$$v = v_T + (v_0 - v_T)\, e^{(-\beta/m)\,t}$$

$$v = \frac{dx}{dt} \qquad dx = v\,dt$$

$$\int_0^x dx = \int_0^t \left(v_T + (v_0 - v_T)\, e^{(-\beta/m)\,t}\right) dt$$

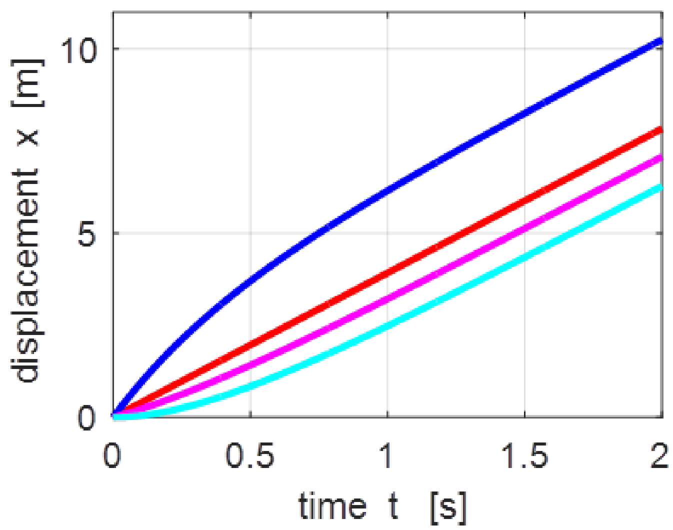
$$x = \left[v_T\, t - \left(\frac{m}{\beta}\right)(v_0 - v_T)\, e^{(-\beta/m)\,t} \right]_0^t \qquad\qquad t \to \infty \qquad\qquad x \to v_T$$

$$x = v_T\, t - \left(\frac{m}{\beta}\right)(v_0 - v_T)\, e^{(-\beta/m)\,t} + \left(\frac{m}{\beta}\right)(v_0 - v_T)$$

$$x = v_T\, t + \left(\frac{m}{\beta}\right)(v_0 - v_T)\left(1 - e^{(-\beta/m)\,t}\right)$$

$$v_0 = 0 \quad \Rightarrow \quad x = v_T\left(t + \left(\frac{m}{\beta}\right)\left(e^{(-\beta/m)\,t} - 1\right)\right)$$

Initial values for velocity v_0 [m.s^{-1}]



blue: 10 red: v_T magenta: 2 cyan: 0

So far we have only considered the case where the initial velocity was either zero or a positive quantity ($v_0 \geq 0$), i.e., the object was released from rest or projected downward. We will now consider the case where the object was project vertically upward ($v_0 < 0$). Note: in our frame of reference, the origin is taken as $x = 0$, the position of the object at time $t = 0$; down is the positive direction and up is the negative direction.

When the object is launched upward at time $t = 0$, the initial velocity has a negative value. Let u be the magnitude of the initial velocity v_0

$$v_0 < 0 \quad v_0 = -u \quad u > 0$$

Therefore, the equation for the velocity v as a function of time t can be expressed as

$$v = v_T + (v_0 - v_T)\, e^{(-\beta/m)\,t}$$
$$v = v_T - (u + v_T)\, e^{(-\beta/m)\,t}$$

We can now find the time t_{up} it takes for the object to rise to its maximum height x_{up} above the origin (remember: up is negative). At the highest point $v = 0$, therefore,

$$0 = v_T - (u + v_T)\, e^{(-\beta/m)\,t_{up}}$$
$$t_{up} = \left(\frac{m}{\beta}\right) \log_e \left(1 + \frac{u}{v_T}\right)$$

The maximum height x_{up} reached by the object in time $t = t_{up}$ is

$$x = v_T\, t + \left(\frac{m}{\beta}\right)\left(v_0 - v_T\right)\left(1 - e^{(-\beta/m)\,t}\right)$$

$$x_{up} = v_T\, t_{up} - \left(\frac{m}{\beta}\right)\left(u + v_T\right)\left(1 - e^{(-\beta/m)\,t_{up}}\right)$$

For the parameters

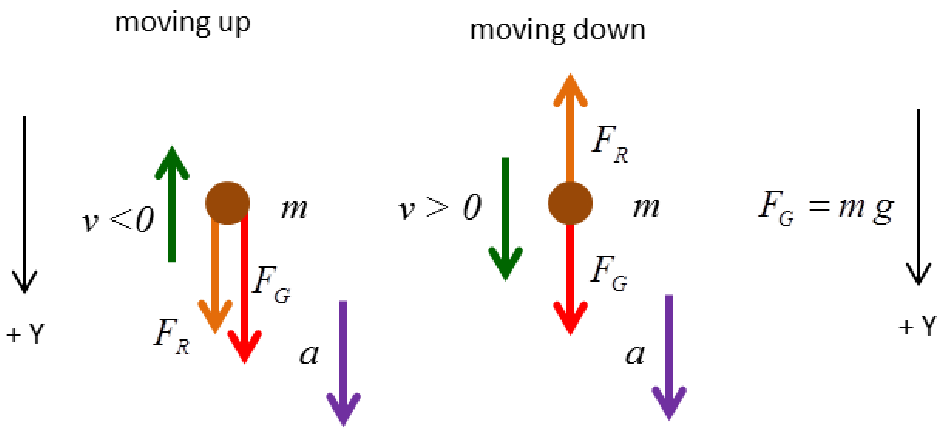
m = 2.00 kg β = 5.00 kg.s^{-1} g = 9.8 m.s^{-2} u = 10 m.s^{-1} v_T = 3.92 m.s^{-1}

The time to reach maximum height is t_{up} = 0.507 s

The max height h_{up} reached is h_{up} = 2.013 m x_{up} = - 2.013 m

A7

The forces acting on the object are the gravitational force F_G (weight) and the resistive force F_R. In our frame of reference, we will take down as the positive direction.



Newton's Second Law

$$m\,a = m\,g + F_R \qquad\qquad m\,a = m\,g - F_R$$

The equation of motion of the object is determined from Newton's Second Law.

$$m\,a = m\,\frac{dv}{dt} = F_G - F_R = mg - \alpha\,v^2\left(v/|v|\right) \qquad a = g - \frac{\alpha}{m}\,v^2\left(v/|v|\right)$$

where a is the acceleration of the object at any instance.

The initial conditions are $t = 0 \quad v = v_0 \quad x = 0 \quad a = g - (\alpha/m)\,v_0^{\,2}\left(\frac{v_0}{|v_0|}\right)$

When a = 0, the velocity is constant $v = v_T$ where v_T is the **terminal velocity**

$$0 = m\,g - \alpha\,v_T^{\,2} \qquad v_T^{\,2} = \frac{m\,g}{\alpha}$$

$$v_T = \sqrt{\frac{m\,g}{\alpha}}$$

We start with the equation of motion then integrate this equation where the limits of the integration are determined by the initial conditions (t = 0 and v = v_0) and final conditions (t and v)

Since the acceleration depends upon v^2 it a more difficult problem then for the linear resistive force example. We have to do separate analytical calculations for the motion when the object is falling or rising.

Velocity of the object is always positive (falling object) $v_0 \geq 0 \quad v \geq 0$

Equation of motion

$$a = g - \frac{\alpha}{m} v^2$$

$$a = \frac{dv}{dt} = g - \left(\frac{\alpha}{m}\right) v^2$$

$$dt = \frac{dv}{g - \left(\frac{\alpha}{m}\right) v^2} = \frac{dv}{\left(\frac{\alpha}{m}\right)\left(\left(\frac{m\,g}{\alpha}\right) - v^2\right)} \qquad {v_T}^2 = \frac{m\,g}{\alpha}$$

$$-\left(\frac{\alpha}{m}\right) dt = \frac{dv}{v^2 - {v_T}^2} = \left(\frac{1}{2v_T}\right)\left(\frac{1}{v - v_T} - \frac{1}{v + v_T}\right) dv$$

$$-\left(2\sqrt{\frac{m\,g}{\alpha}}\right)\left(\frac{\alpha}{m}\right) dt = \left(\frac{1}{v - v_T} - \frac{1}{v + v_T}\right) dv$$

$$-\sqrt{\frac{4\,\alpha\,g}{m}}\, dt = \left(\frac{1}{v - v_T} - \frac{1}{v + v_T}\right) dv$$

$$-\sqrt{\frac{4\,\alpha\,g}{m}} \int_0^t dt = \int_{v_0}^{v} \left(\frac{1}{v - v_T} - \frac{1}{v + v_T}\right) dv$$

$$-\sqrt{\frac{4\,\alpha\,g}{m}}\, t = \left[-\log_e\left(v - v_T\right) - \log_e\left(v + v_T\right)\right]_{v_0}^{v}$$

$$\sqrt{\frac{4\alpha g}{m}}t = \left[\log_e\left(v - v_T\right) + \log_e\left(v + v_T\right)\right]_{v_0}^{v}$$

$$\sqrt{\frac{4\alpha g}{m}}t = \left[\log_e\left(\frac{v - v_T}{v_0 - v_T}\right) + \log_e\left(\frac{v - v_T}{v_0 - v_T}\right)\right]$$

$$-\sqrt{\frac{4\alpha g}{m}}t = \log_e\left\{\left(\frac{v - v_T}{v_0 - v_T}\right)\left(\frac{v_0 + v_T}{v + v_T}\right)\right\}$$

$$\left(\frac{v - v_T}{v_0 - v_T}\right)\left(\frac{v_0 + v_T}{v + v_T}\right) = e^{-\sqrt{\frac{4\alpha g}{m}}t} \qquad \sqrt{\frac{4\alpha g}{m}} = \sqrt{\frac{4\alpha g^2}{mg}} = \sqrt{\frac{4 g^2}{v_T{}^2}} = \frac{2g}{v_T}$$

$$v - v_T = \left(v + v_T\right)\left(\frac{v_0 - v_T}{v_0 + v_T}\right)e^{-\frac{2g}{v_T}t} = \left(v + v_T\right)K \qquad K = \left(\frac{v_0 - v_T}{v_0 + v_T}\right)e^{-\frac{2g}{v_T}t}$$

$$v\left(1 - K\right) = v_T\left(1 + K\right) \qquad v = v_T\left(\frac{1 + K}{1 - K}\right)$$

$$v = v_T\left(\frac{1 + \left(\frac{v_0 - v_T}{v_0 + v_T}\right)e^{-\frac{2g}{v_T}t}}{1 - \left(\frac{v_0 - v_T}{v_0 + v_T}\right)e^{-\frac{2g}{v_T}t}}\right)$$

$$v = v_T\left(\frac{\left(v_0 + v_T\right) + \left(v_0 - v_T\right)e^{-\frac{2g}{v_T}t}}{\left(v_0 + v_T\right) - \left(v_0 - v_T\right)e^{-\frac{2g}{v_T}t}}\right)$$

valid only if $v_0 \geq 0 \quad v \geq 0$

We can now calculate the displacement *x* as a function of velocity *v*

$$a = \frac{dv}{dt} = \frac{v\,dv}{dx} = g - (\alpha / m) v^2$$

$$\frac{v\,dv}{dx} = (\alpha / m)\left(m\,g / \alpha - v^2\right) \qquad {v_T}^2 = m\,g / \alpha$$

$$dx = \left(\frac{m}{\alpha}\right)\frac{v\,dv}{\left({v_T}^2 - v^2\right)} = \left(\frac{-{V_T}^2}{2\,g}\right)\frac{(-2v)\,dv}{\left({v_T}^2 - v^2\right)}$$

$$\int_0^x dx = \left(\frac{-{v_T}^2}{2\,g}\right)\int_{v_0}^{v} \frac{(-2v)}{\left({v_T}^2 - v^2\right)} dv$$

$$x = \left(\frac{-{v_T}^2}{2\,g}\right)\left[\log_e\left({v_T}^2 - v^2\right)\right]_{v_0}^{v}$$

$$x = \left(\frac{{v_T}^2}{2\,g}\right)\log_e\left(\frac{{v_T}^2 - {v_0}^2}{{v_T}^2 - v^2}\right)$$

valid only if $v_0 \ge 0 \quad v \ge 0$

We can now investigate what happens as time $t \to \infty$

$$v(t \to \infty) = v_T \left(\frac{(v_0 + v_T) + 0}{(v_0 + v_T) - 0} \right) \qquad e^{-\frac{2g}{v_T}t} \to 0$$

$$v(t \to \infty) = v_T$$

In falling, the object will finally reach a constant velocity v_T ($a = 0$) which is known as the terminal velocity.

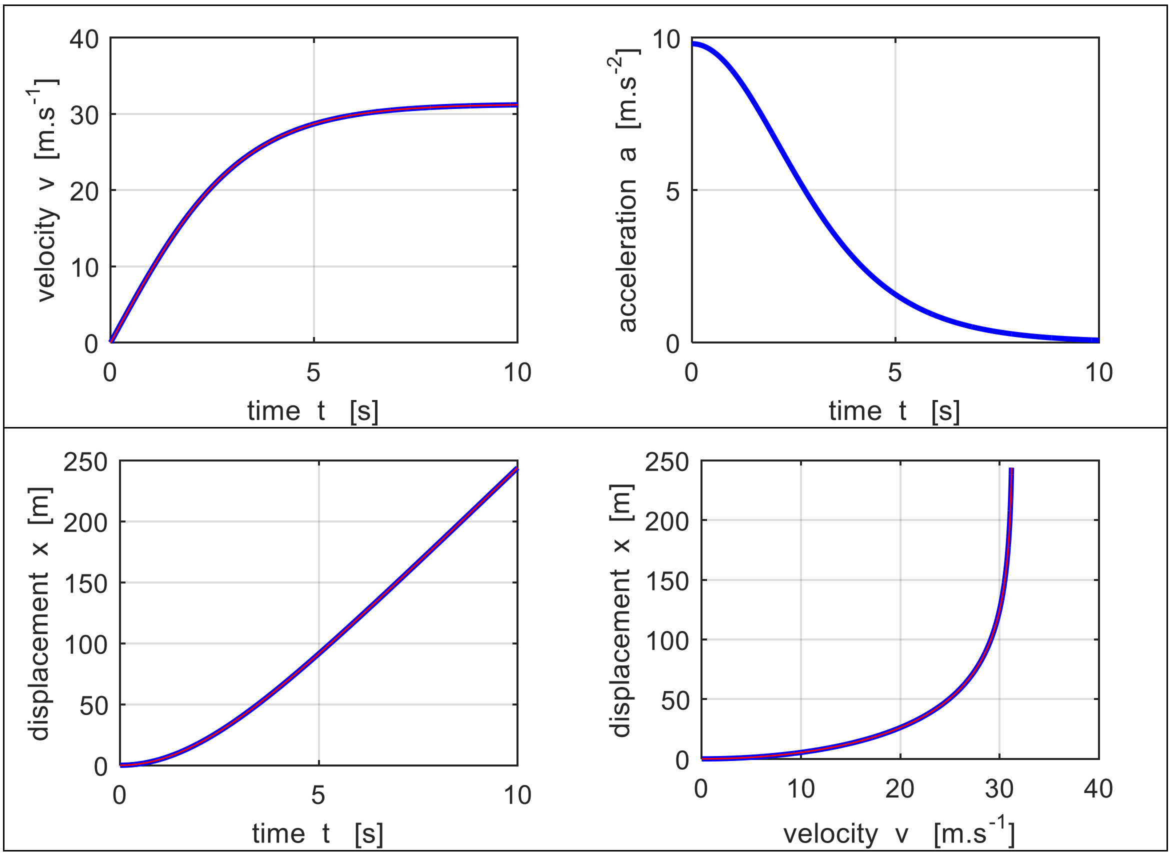
$$t \to \infty \quad v \to v_T \quad v_T - v \to 0 \quad \frac{1}{v_T - v} \to \infty$$

$$x = \left(\frac{V_T^{\,2}}{2g} \right) \log_e \left(\frac{v_T^{\,2} - v_0^{\,2}}{v_T^{\,2} - v^2} \right) \to \infty$$

In falling, as time *t* increases the objects displacement *x* just gets larger and larger.

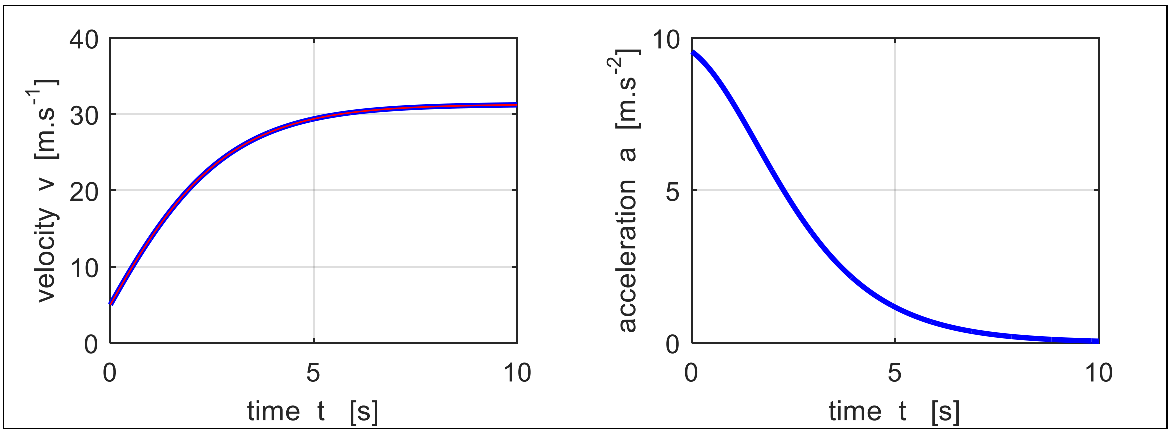
Example Small rock dropped from rest:

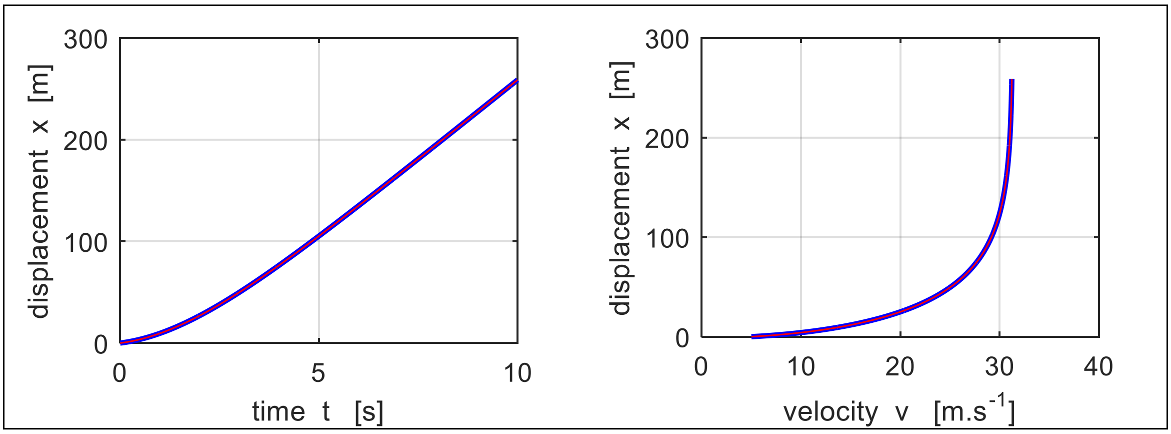
$m = 0.010$ kg $\alpha = 1.00\times10^{-4}$ kg.m^{-1} $v_0 = 0$ m.s^{-1} $\Rightarrow$ $v_T = 31.3$ m.s^{-1}



Example Small rock thrown vertically downward ($v < v_T$)

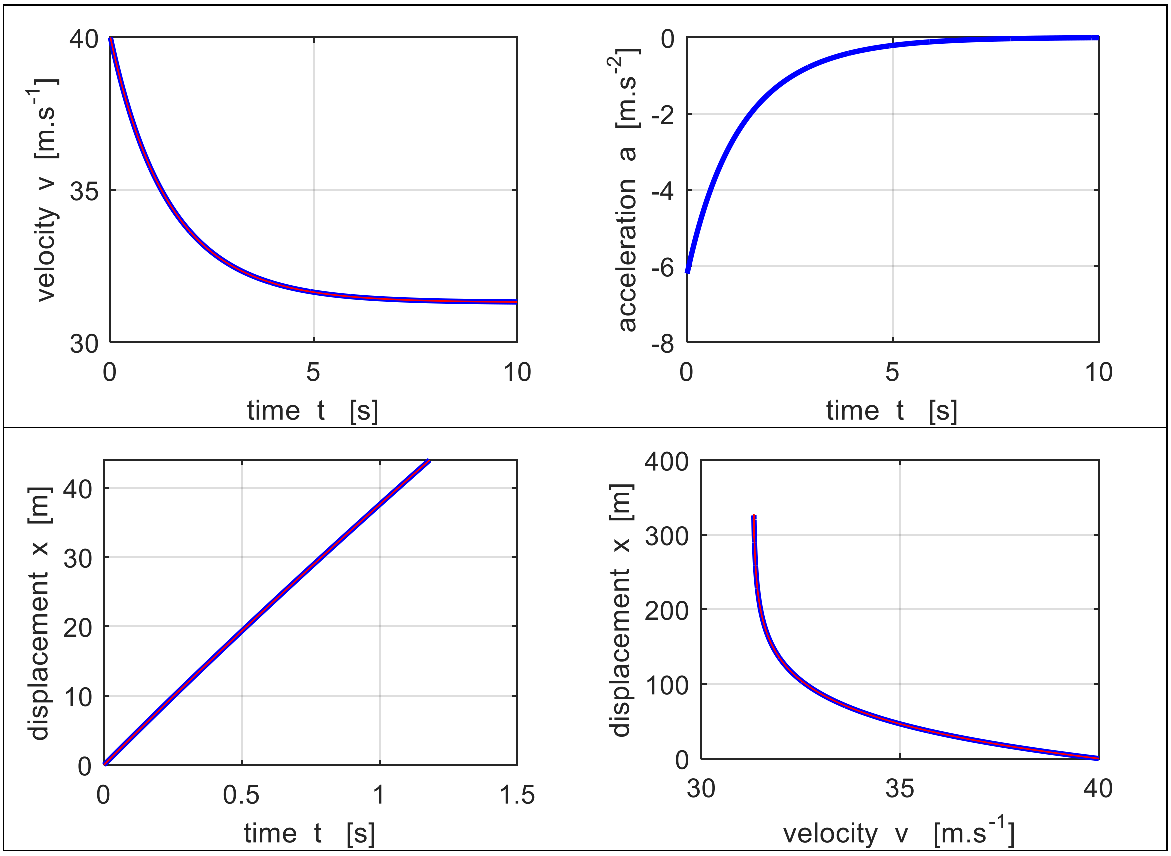
$m = 0.010$ kg $\alpha = 1.00\times10^{-4}$ kg.m^{-1} $v_0 = +5.00$ m.s^{-1} $\Rightarrow$ $v_T = 31.3$ m.s^{-1}





Example Small rock thrown vertically downward ($v > v_T$)

$$m = 0.010\ \text{kg} \quad \alpha = 1.00\times10^{-4}\ \text{kg.m}^{-1} \quad v_0 = +40.0\ \text{m.s}^{-1} \quad \Rightarrow \quad v_T = 31.3\ \text{m.s}^{-1}$$



For problems in which the object is projected vertically upward, you have to divide the problem into two parts. (1) Calculate the time to reach its maximum height and calculate the maximum height reached for the upward motion. (2) Reset the initial conditions to the position at maximum height where the initial velocity becomes $v_0 = 0$ and do the calculations for the downward movement of the object.

Velocity of the object negative and moving up $v_0 < 0 \quad and \quad v < 0$

Equation of motion

$$a = g + \frac{\alpha}{m} v^2 \qquad \text{valid only if} \quad v_0 < 0 \quad and \quad v < 0$$

Note: up is the positive direction

$$a = \frac{dv}{dt} = g + \left(\frac{\alpha}{m}\right) v^2$$

$$dt = \frac{dv}{g + \left(\frac{\alpha}{m}\right) v^2} = \frac{dv}{\left(\frac{\alpha}{m}\right)\left(\left(\frac{m\,g}{\alpha}\right) + v^2\right)} \qquad {v_T}^2 = \frac{m\,g}{\alpha}$$

$$\left(\frac{\alpha}{m}\right) dt = \frac{dv}{v^2 + {v_T}^2}$$

$$\left(\frac{\alpha}{m}\right) \int_0^t dt = \int_{v_0}^{v} \frac{dv}{v^2 + {v_T}^2}$$

$$\text{Standard Integral} \qquad \int \frac{dx}{a^2 + x^2} = \left(\frac{1}{a}\right) \text{atan}\left(\frac{x}{a}\right) + C$$

$$\left(\frac{\alpha}{m}\right) t = \left(\frac{1}{v_T}\right)\left[\text{atan}\left(\frac{v}{v_T}\right)\right]_{v_0}^{v} \qquad \left(\frac{m}{\alpha\, v_T}\right) = \left(\frac{m}{\alpha\, v_T}\frac{g}{g}\right) = \left(\frac{v_T}{g}\right)$$

$$t = \left(\frac{v_T}{g}\right)\left[\text{atan}\left(\frac{v}{v_T}\right)\right]_{v_0}^{v} = \left(\frac{v_T}{g}\right)\left[\text{atan}\left(\frac{v}{v_T}\right) - \text{atan}\left(\frac{v_0}{v_T}\right)\right]$$

The time tup to reach maximum height occurs when $v = 0$

$$t_{up} = \left(\frac{v_T}{g}\right)\left[\text{atan}\left(\frac{0}{v_T}\right) - \text{atan}\left(\frac{v_0}{v_T}\right)\right] = -\left(\frac{v_T}{g}\right) \text{atan}\left(\frac{v_0}{v_T}\right) \qquad v_0 < 0$$

The velocity v as a function of time t is

$$\operatorname{atan}\left(\frac{v}{v_T}\right) = \operatorname{atan}\left(\frac{v_0}{v_T}\right) + \left(\frac{g}{v_T}\right)t$$

$$v = v_T \tan\left[\operatorname{atan}\left(\frac{v_0}{v_T}\right) + \left(\frac{g}{v_T}\right)t\right] \qquad \operatorname{atan}\theta \equiv \tan^{-1}\theta$$

$$v_0 < 0 \quad and \quad v < 0$$

The displacement x as a function of velocity v is

$$a = \frac{dv}{dt} = \frac{v\,dv}{dx} = g + (\alpha/m)v^2$$

$$\frac{v\,dv}{dx} = (\alpha/m)\left(m\,g/\alpha - v^2\right) \qquad v_T^{\ 2} = m\,g/\alpha$$

$$dx = \left(\frac{m}{\alpha}\right)\frac{v\,dv}{\left(v_T^{\ 2} + v^2\right)} = \left(\frac{v_T^{\ 2}}{2\,g}\right)\frac{(2v)\,dv}{\left(v_T^{\ 2} + v^2\right)}$$

$$\int_0^x dx = \left(\frac{v_T^{\ 2}}{2\,g}\right)\int_{v_0}^{v} \frac{(2v)}{\left(v_T^{\ 2} + v^2\right)}dv$$

$$v_0 < 0 \quad and \quad v < 0$$

$$x = \left(\frac{v_T^{\ 2}}{2\,g}\right)\left[\log_e\left(v_T^{\ 2} + v^2\right)\right]_{v_0}^{v}$$

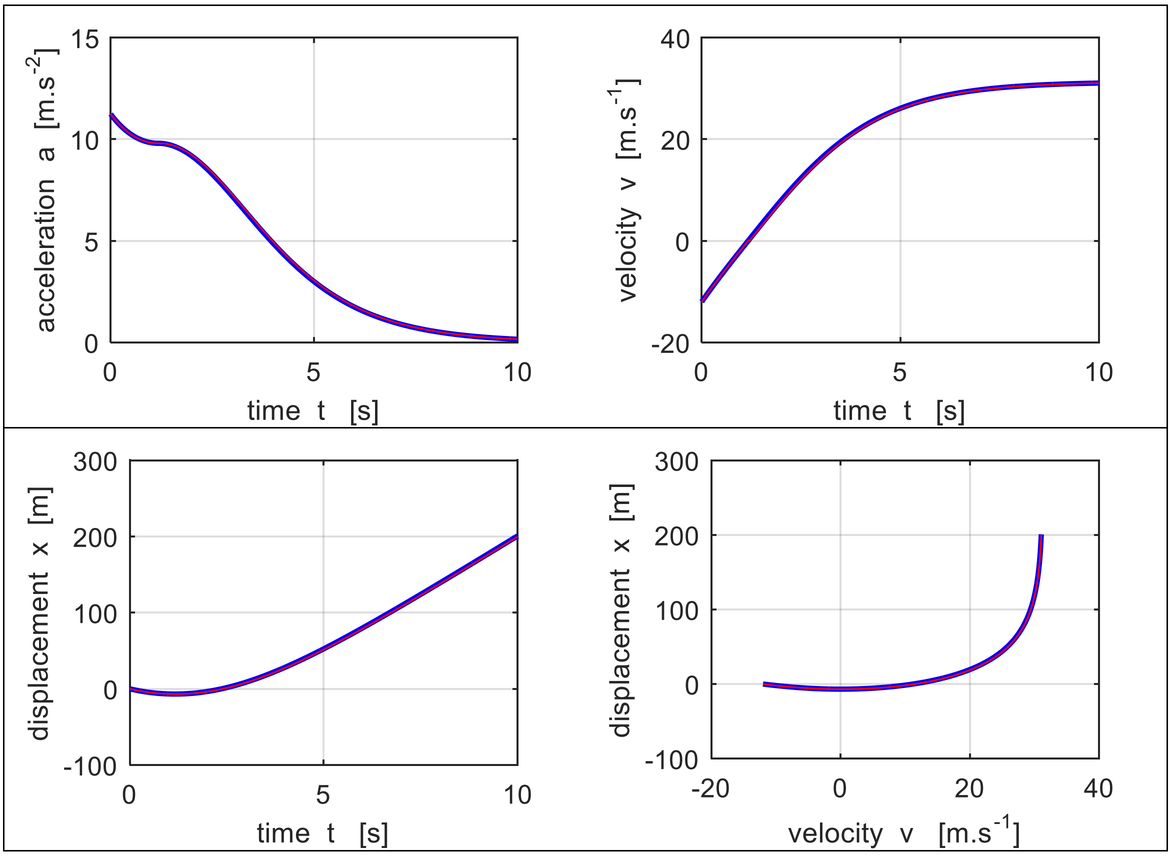
$$x = \left(\frac{v_T^{\ 2}}{2\,g}\right)\log_e\left(\frac{v_T^{\ 2} + v^2}{v_T^{\ 2} + v_0^{\ 2}}\right)$$

The maximum height x_{up} reached by the object occurs when $v = 0$

$$x_{up} = \left(\frac{v_T^{\ 2}}{2\,g}\right)\log_e\left(\frac{v_T^{\ 2}}{v_T^{\ 2} + v_0^{\ 2}}\right)$$

Example Small rock thrown vertically upward ($v_0 < 0$ $v_0 = -u$ $u > 0$)

$m = 0.010$ kg $\alpha = 1.00\times10^{-4}$ kg.m^{-1} $v_0 = -12.0$ m.s^{-1} $\Rightarrow$ $v_T = 31.3$ m.s^{-1}



The terminal velocity v_T is

$$v_T^{\ 2} = m\,g\,/\,\alpha$$

$$v_T = \sqrt{m\,g\,/\,\alpha} = \sqrt{\left(10^{-2}\right)(9.8)/\left(10^{-4}\right)}\ \text{m.s}^{-1}$$

$$v_T = 31.31\ \ \text{m.s}^{-1}$$

When $v = 0$ the object reaches its maximum height x_{up} (up is negative)

$$x_{up} = \left(\frac{v_T^{\ 2}}{2\,g}\right)\log_e\left(\frac{v_T^{\ 2}}{v_T^{\ 2}+v_0^{\ 2}}\right)$$

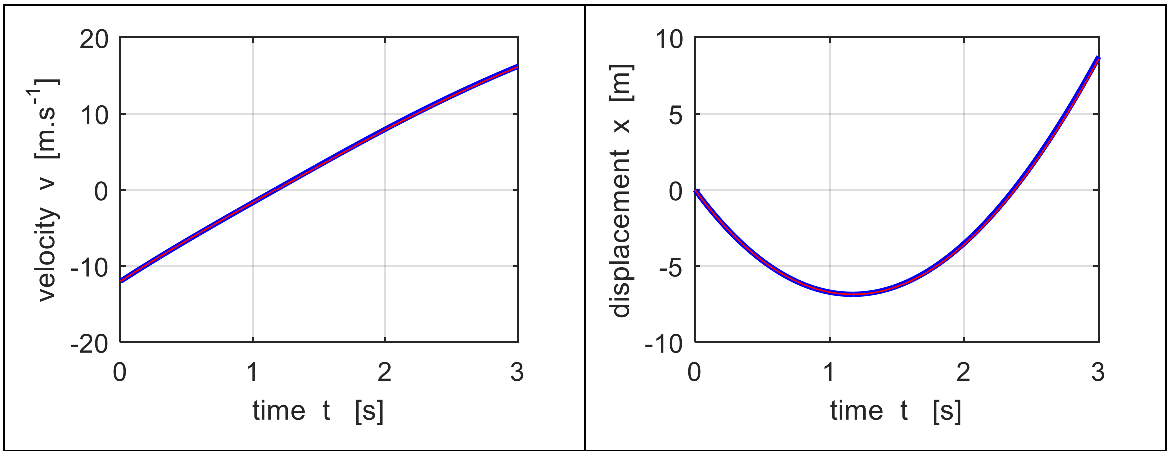
$$x_{up} = -6.855\ \text{m}$$

The time t_{up} to reach the maximum height

$$t_{up} = -\left(\frac{v_T}{g}\right)\mathrm{atan}\left(\frac{v_0}{v_T}\right)$$

$$t_{up} = 1.169\ \mathrm{s}$$

The calculations agree with the values for t_{up} and x_{up} determined from the graphs.



From the graphs:

$x = 0$ time $t = 2.275$ s velocity $v = 11.21\ \mathrm{m.s^{-1}}$

Time to fall from max height to origin $x = 0$ t_{down} = (2.375 – 1.169) s = 1.206 s

takes slight longer to fall then rise to and from origin to max height

Launch speed = 12.00 $\mathrm{m.s^{-1}}$ slightly greater than return speed = 11.21 $\mathrm{m.s^{-1}}$

In the absence of any resistive forces $a = g$ $v = v_0 + a\,t$ $v^2 = {v_0}^2 + 2\,a\,s$

At maximum height $v = 0$ $t_{up} = 1.2245\,\mathrm{s}$ $x_{up} = -7.3469\,\mathrm{m}$