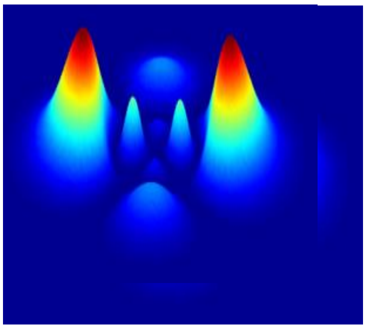

ADVANCED HIGH SCHOOL MATHEMATICS

INTEGRATION

EXERCISES

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To find a Question or Answer use the find function: *control f*

Question 4 **Q4**

Answer 15 **A15**

QUESTIONS

Evaluate the following integrals

Q1 $$I = \int \left(\frac{1+x}{x}\right)^2 dx$$

Q2 $$I = \int \left(\frac{x^2}{x^2+1}\right) dx$$

Q3 $$I = \int x\left(1+x^2\right)^4 dx$$

Q4 $$I = \int \left(\frac{x}{\sqrt{1-x}} \right) dx$$

Q5 $$I = \int_0^{\pi/4} \sin^2(2x)\, dx$$

Q6 $$I = \int \sin^2(x)\cos(x)\, dx$$

Q7 $$I = \int \sin^2(x)\cos^3(x)\, dx$$

Q8 $$I = \int \frac{dx}{a^2 + x^2}$$

Q9 $$\int \sin^{-1} x \, dx$$

Q10 $$I = \int \left(\frac{1+x}{x} \right)^2 dx$$

Q11 $$I = \int \left(\frac{x^2}{x^2+1} \right) dx$$

Q12 $$I = \int x\left(1+x^2\right)^4 dx$$

Q13 $$I = \int \left(\frac{x}{\sqrt{1-x}} \right) dx$$

Q14 $$I = \int_0^{\pi/4} \sin^2(2x)\, dx$$

Q15 $$I = \int \sin^2(x)\cos(x)\, dx$$

Q16 $I = \int \sin^2(x)\cos^3(x)\,dx$

Q17 $I = \int \dfrac{dx}{a^2 + x^2}$

Q18 $I = \int \sin^{-1} x\,dx$

Q19 $I = \int e^{3\theta}\cos(4\theta)\,d\theta$

Q20 $I = \int x^n \log_e(x)\,dx$

Q21 $I = \int x^n e^x\,dx$ and evaluate the integral when n = 3

Q22 $I = \int \cos^n x\,dx$ and evaluate the integral when n = 4

Q23 $I = \int \dfrac{dx}{x^2 - 4x - 1}$

Q24 $I = \int \dfrac{dx}{3x^2 + 6x + 10}$

Q25 $I = \int \dfrac{3x + 2}{x^2 - 4x + 1}\,dx$

Q26 $$\int_1^{\sqrt{3}} \frac{1+x}{x^2\left(1+x^2\right)}dx$$

Q26 $$\int_0^1 \left(e^x-1\right)^{1/2} dx$$

Q28 $$I=\int \frac{dx}{x^2\sqrt{1+x}}$$

$$I=\int \frac{9x-2}{2x^2-7x+3}dx$$

Q29

$$I=\int \frac{3x^2-2x+1}{\left(x^2+1\right)\left(x^2+2\right)}dx$$

Q30

$$I=\int \frac{2x^2+3x-1}{x^3-x^2+x-1}dx$$

Q31

ANSWERS

A1

$$I = \int \left(\frac{1+x}{x}\right)^2 dx$$

$$I = \int \left(\frac{1+x}{x}\right)^2 dx = I = \int \left(1 + \frac{2}{x} + \frac{1}{x^2}\right) dx$$

$$I = x + 2\log_e(x) - \frac{1}{x} + K$$

A2

$$I = \int \left(\frac{x^2}{x^2+1}\right) dx$$

$$N = \frac{x^2}{x^2+1} = A + \frac{B}{x^2+1} = \frac{Ax^2 + A + B}{x^2+1}$$

$$A = 1 \quad B = -1$$

$$I = \int \left(1 - \frac{1}{x^2+1}\right) dx$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$I = x - \tan^{-1}(x) + K$$

A3

$$I = \int x\left(1 + x^2\right)^4 dx$$

$$u = 1 + x^2 \quad du = 2x\,dx \quad x\,dx = \frac{u}{2}$$

$$I = \left(\frac{1}{2}\right)\int u^4\,du$$

$$I = \left(\frac{1}{2}\right)\left(\frac{1}{5}\right)u^5 + K$$

$$I = \left(\frac{1}{10}\right)\left(1 + x^2\right)^5 + K$$

A4

$$I = \int \left(\frac{x}{\sqrt{1 - x}}\right)dx$$

$$u = \sqrt{1 - x} \quad u^2 = 1 - x \quad x = 1 - u^2 \quad dx = -2u\,du$$

$$I = -2\int \left(\frac{1 - u^2}{u}\right)u\,du = -2\int\left(1 - u^2\right)du$$

$$I = -2\left(u - u^3/3\right) + K =$$

$$I = -(2/3)u\left(3 - u^2\right) + K$$

$$I = -(2/3)\sqrt{1 - x}\,(3 - 1 + x) + K$$

$$I = -(2/3)\sqrt{1 - x}\,(2 + x) + K$$

A5

$$I = \int_0^{\pi/4} \sin^2(2x)\,dx$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x)$$

$$\sin^2(2x) = \frac{1}{2}\left(1 - \cos(4x)\right)$$

$$I = \frac{1}{2}\int_0^{\pi/4} \left(1 - \cos(4x)\right)dx$$

$$I = \frac{1}{2}\left[x - \tfrac{1}{4}\sin(4x)\right]_0^{\pi/4}$$

$$I = \pi/8$$

A6

$$I = \int \sin^2(x)\cos(x)\,dx$$

$$I = \frac{1}{3}\sin^3(x) + K$$

A7

$$I = \int \sin^2(x)\cos^3(x)\,dx$$

$$I = \int \cos(x)\sin^2(x)\cos^2(x)\,dx \qquad \sin^2(x) + \cos^2(x) = 1$$

$$I = \int \cos(x)\sin^2(x)\left(1 - \sin^2(x)\right)dx$$

$$I = \int \left(\cos(x)\sin^2(x) - \cos(x)\sin^4(x)\right)dx$$

$$I = \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + K$$

A8

$$I = \int \frac{dx}{a^2 + x^2}$$

$$x = a\tan\theta \quad dx = a\sec^2\theta\, d\theta \quad \theta = \tan^{-1}\left(\frac{x}{a}\right)$$

$$I = \int \frac{a\sec^2\theta\, d\theta}{a^2 + a^2\tan^2\theta} = \frac{1}{a}\int d\theta$$

$$I = \frac{\theta}{a}$$

$$I = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + K$$

A9

$$I = \int \sin^{-1} x\, dx$$

$$\theta = \sin^{-1} x \quad \sin\theta = x \quad \cos\theta\, d\theta = dx$$

$$I = \int \theta\cos\theta\, d\theta$$

integrate by parts

$$u = \theta \quad du = d\theta \quad dv = \cos\theta\, d\theta \quad v = \sin\theta$$

$$\int u\, dv = uv - \int v\, du$$

$$I = \theta\sin\theta - \int \sin\theta\, d\theta$$

$$I = \theta\sin\theta + \cos\theta + K$$

$$\sin\theta = x \quad \sin^2\theta + \cos^2\theta = 1 \quad \cos^2\theta = 1 - \sin^2\theta \quad \cos\theta = \sqrt{1 - x^2}$$

$$I = x\sin^{-1} x + \sqrt{1 - x^2} + K$$

A10

$$I = \int \left(\frac{1+x}{x}\right)^2 dx$$

$$I = \int \left(\frac{1+x}{x}\right)^2 dx = I = \int \left(1+\frac{2}{x}+\frac{1}{x^2}\right)dx$$

$$I = x + 2\log_e(x) - \frac{1}{x} + K$$

A11

$$I = \int \left(\frac{x^2}{x^2+1}\right)dx$$

$$N = \frac{x^2}{x^2+1} = A + \frac{B}{x^2+1} = \frac{Ax^2+A+B}{x^2+1}$$

$$A=1 \quad B=-1$$

$$I = \int \left(1-\frac{1}{x^2+1}\right)dx$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$I = x - \tan^{-1}(x) + K$$

A12

$$I = \int x\left(1+x^2\right)^4 dx$$

$$u = 1 + x^2 \quad du = 2x\,dx \quad x\,dx = \frac{u}{2}$$

$$I = \left(\frac{1}{2}\right)\int u^4\,du$$

$$I = \left(\frac{1}{2}\right)\left(\frac{1}{5}\right)u^5 + K$$

$$I = \left(\frac{1}{10}\right)\left(1+x^2\right)^5 + K$$

A13

$$I = \int \left(\frac{x}{\sqrt{1-x}}\right)dx$$

$$u = \sqrt{1-x} \quad u^2 = 1 - x \quad x = 1 - u^2 \quad dx = -2u\,du$$

$$I = -2\int \left(\frac{1-u^2}{u}\right)u\,du = -2\int\left(1-u^2\right)du$$

$$I = -2\left(u - u^3/3\right) + K =$$

$$I = -(2/3)u\left(3-u^2\right) + K$$

$$I = -(2/3)\sqrt{1-x}\,(3-1+x) + K$$

$$I = -(2/3)\sqrt{1-x}\,(2+x) + K$$

A14

$$I = \int_0^{\pi/4} \sin^2(2x)\,dx$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x)$$

$$\sin^2(2x) = \frac{1}{2}\left(1 - \cos(4x)\right)$$

$$I = \frac{1}{2}\int_0^{\pi/4} \left(1 - \cos(4x)\right)dx$$

$$I = \frac{1}{2}\left[x - \tfrac{1}{4}\sin(4x)\right]_0^{\pi/4}$$

$$I = \pi / 8$$

A15

$$I = \int \sin^2(x)\cos(x)\,dx$$

$$I = \frac{1}{3}\sin^3(x) + K$$

A16

$$I = \int \sin^2(x)\cos^3(x)\,dx$$

$$I = \int \cos(x)\sin^2(x)\cos^2(x)\,dx \qquad \sin^2(x) + \cos^2(x) = 1$$

$$I = \int \cos(x)\sin^2(x)\left(1 - \sin^2(x)\right)dx$$

$$I = \int \left(\cos(x)\sin^2(x) - \cos(x)\sin^4(x)\right)dx$$

$$I = \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + K$$

A17

$$I = \int \frac{dx}{a^2 + x^2}$$

$$x = a\tan\theta \quad dx = a\sec^2\theta\, d\theta \quad \theta = \tan^{-1}\left(\frac{x}{a}\right)$$

$$I = \int \frac{a\sec^2\theta\, d\theta}{a^2 + a^2\tan^2\theta} = \frac{1}{a}\int d\theta$$

$$I = \frac{\theta}{a}$$

$$I = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + K$$

A18

$$I = \int \sin^{-1} x\, dx$$

$$\theta = \sin^{-1} x \quad \sin\theta = x \quad \cos\theta\, d\theta = dx$$

$$I = \int \theta \cos\theta\, d\theta$$

integrate by parts

$$u = \theta \quad du = d\theta \quad dv = \cos\theta\, d\theta \quad v = \sin\theta$$

$$\int u\, dv = uv - \int v\, du$$

$$I = \theta\sin\theta - \int \sin\theta\, d\theta$$

$$I = \theta\sin\theta + \cos\theta + K$$

$$\sin\theta = x \quad \sin^2\theta + \cos^2\theta = 1 \quad \cos^2\theta = 1 - \sin^2\theta \quad \cos\theta = \sqrt{1 - x^2}$$

$$I = x\sin^{-1} x + \sqrt{1 - x^2} + K$$

A19

$$I = \int e^{3\theta} \cos(4\theta)\, d\theta$$

Integrate by parts $\int u\, dv = u\, v - \int v\, du \quad u = e^{3\theta}$

$du = 3e^{3\theta}\, d\theta \qquad dv = \cos(4\theta)\, d\theta \quad v = \left(\frac{1}{4}\right)\sin(4\theta)$

$$I = \left(\frac{1}{4}\right)e^{3\theta}\sin(4\theta) - \left(\frac{3}{4}\right)\int e^{3\theta}\sin(4\theta)\, d\theta$$

$$I_1 = \int e^{3\theta}\sin(4\theta)\, d\theta$$

Integrate by parts $\int u\, dv = u\, v - \int v\, du \quad u = e^{3\theta}$

$du = 3e^{3\theta}\, d\theta \qquad dv = \sin(4\theta)\, d\theta \quad v = \left(\frac{-1}{4}\right)\cos(4\theta)$

$$I_1 = \left(\frac{-1}{4}\right)e^{3\theta}\cos(4\theta) + \left(\frac{3}{4}\right)\int 3e^{3\theta}\cos(4\theta)\, d\theta$$

$$= \left(\frac{-1}{4}\right)e^{3\theta}\cos(4\theta) + \left(\frac{3}{4}\right)I$$

$$I = \left(\frac{1}{4}\right)e^{3\theta}\sin(4\theta) - \left(\frac{3}{4}\right)\left(\left(\frac{-1}{4}\right)e^{3\theta}\cos(4\theta) + \left(\frac{3}{4}\right)I\right)$$

$$I = e^{3\theta}\left(\left(\frac{1}{4}\right)\sin(4\theta) + \left(\frac{3}{4^2}\right)\cos(4\theta)\right) - \left(\frac{3}{4}\right)^2 I$$

$$\left(1 + \left(\frac{3}{4}\right)^2\right)I = \left(\frac{1}{4}\right)e^{3\theta}\left(\sin(4\theta) + \left(\frac{3}{4}\right)\cos(4\theta)\right)$$

$$\left(\frac{4^2 + 3^2}{4^2}\right)I = \left(\frac{e^{3\theta}}{4}\right)\left(\sin(4\theta) + \left(\frac{3}{4}\right)\cos(4\theta)\right)$$

$$I = \left(\frac{1}{4^2 + 3^2}\right)e^{3\theta}\left(4\sin(4\theta) + 3\cos(4\theta)\right)$$

$$= \frac{e^{3\theta}}{25}\left(4\sin(4\theta) + 3\cos(4\theta)\right)$$

A20

$$I = \int x^n \log_e(x)\,dx \qquad \log_e(x) \equiv \ln(x)$$

Integrate by parts $\int u\,dv = u\,v - \int v\,du$

$$u = \log_e(x) \quad du = \frac{dx}{x} \qquad dv = x^n \quad v = \frac{1}{n+1}x^{n+1}$$

$$I = \frac{1}{n+1}x^{n+1}\log_e(x) - \frac{1}{n+1}\int x^n\,dx$$

$$I = \frac{x^{n+1}}{n+1}\log_e(x) - \frac{x^{n+1}}{(n+1)^2} + K$$

$$I = \frac{x^{n+1}}{(n+1)^2}\left((n+1)\log_e(x) - 1\right) + K$$

A21

$$I_n = \int x^n\,e^x\,dx$$

Integrate by parts $\int u\,dv = u\,v - \int v\,du$

$$u = x^n \quad du = n\,x^{n-1}\,dx \qquad dv = e^x \quad v = e^x$$

$$I_n = x^n\,e^x - n\int x^{n-1}\,e^x dx$$

$$I_n = x^n\,e^x - n\,I_{n-1}$$

$$I_0 = \int e^x\,dx = e^x$$

$$I_1 = x\,e^x - e^x = e^x(x-1)$$

$$I_2 = x^2\,e^x - 2e^x(x-1) = e^x\left(x^2 - 2x + 2\right)$$

$$I_3 = x^3\,e^x - 3e^x\left(x^2 - 2x + 2\right) = e^x\left(x^3 - 3x^2 + 6x - 6\right)$$

A22

$$I_n = \int \cos^n x \, dx$$

integrate by parts $\int u \, dv = u v - \int v \, du$

$$u = \cos^{n-1} x \quad du = -(n-1) \sin x \cos^{n-2} x \, dx$$

$$dv = \cos x \, dx \quad v = \sin x$$

$$I_n = \sin x \cos^{n-1} x + (n-1) \int \sin^2 x \cos^{n-2} x \, dx$$

$$\sin^2 x + \cos^2 x = 1 \quad \sin^2 x = 1 - \cos^2 x$$

$$I_n = \sin x \cos^{n-1} x + (n-1) \int \left(\cos^{n-2} x - \cos^n x\right) dx$$

$$I_n = \sin x \cos^{n-1} x + (n-1) \left(I_{n-2} - I_n\right) + K$$

$$I_n \left(1 + n - 1\right) = \sin x \cos^{n-1} x + (n-1) \, I_{n-2}$$

$$I_n = \left(\frac{1}{n}\right) \sin x \cos^{n-1} x + \left(\frac{n-1}{n}\right) I_{n-2}$$

$$n = 4$$

$$I_4 = \int \cos^4 x \, dx$$

$$I_4 = \left(\frac{1}{4}\right) \sin x \cos^3 x + \left(\frac{3}{4}\right) I_2 + K$$

$$I_2 = \left(\frac{1}{2}\right) \sin x \cos x + \left(\frac{1}{2}\right) I_0 \quad I_0 = x$$

$$I_4 = \left(\frac{1}{4}\right) \sin x \cos^3 x + \left(\frac{3}{4}\right) \left(\left(\frac{1}{2}\right) \sin x \cos x + \left(\frac{1}{2}\right) x \right) + K$$

$$I_4 = \left(\frac{1}{4}\right) \sin x \cos^3 x + \left(\frac{3}{8}\right) \left(\sin x \cos x + x\right) + K$$

$$I_4 = \left(\frac{1}{8}\right) 2 \sin x \cos x \cos^2 x + \left(\frac{3}{16}\right) \left(2 \sin x \cos x + 2x\right) + K$$

$$I_4 = \left(\frac{1}{8}\right) \sin(2x) \cos^2 x + \left(\frac{3}{16}\right) \sin(2x) + \left(\frac{3x}{8}\right) + K$$

A23

$$I = \int \frac{dx}{x^2 - 4x - 1}$$

let $x^2 - 4x - 1 = (x - A)^2 + B = x^2 - 2Ax + A^2 + B$

$A = 2 \quad 4 + B = -1 \quad B = -5 \quad a = \sqrt{5} \quad B = -a^2$

$z = x - A = x - 2 \quad dx = dz$

$$\frac{1}{x^2 - 4x - 1} = \frac{1}{z^2 - a^2} = \frac{1}{2a}\left(\frac{1}{z-a} - \frac{1}{z+a}\right)$$

$$I = \frac{1}{2a}\int\left(\frac{1}{z-a} - \frac{1}{z+a}\right)dz$$

$$I = \frac{1}{2a}\left(\log_e(z-a) - \log_e(z+a)\right) + K$$

$$I = \frac{1}{2\sqrt{5}}\left(\log_e\left(\frac{x - 2 - \sqrt{5}}{x - 2 + \sqrt{5}}\right)\right) + K$$

A24

$$I = \int \frac{dx}{3x^2 + 6x + 10}$$

let $3x^2 + 6x + 10 = 3\left(x^2 + 2x + 10/3\right) = 3\left(x^2 + 2x + 1 + 10/3 - 1\right)$

$$3x^2 + 6x + 10 = 3\left((x+1)^2 + 7/3\right)$$

$$z = x + 1 \qquad dx = dz \qquad a = \sqrt{7/3}$$

$$I = \frac{1}{3}\int \frac{dx}{z^2 + a^2}$$

$$I = \frac{1}{3a}\tan^{-1}\left(\frac{z}{a}\right) + K$$

$$I = \frac{1}{\sqrt{21}}\tan^{-1}\left(\sqrt{\frac{3}{7}}(x+1)\right) + K$$

$$I = \frac{\sqrt{21}}{21}\tan^{-1}\left(\frac{\sqrt{21}}{7}(x+1)\right) + K$$

A25

$$I = \int \frac{3x+2}{x^2-4x+1}\,dx$$

let $\quad y = x^2-4x+1 \qquad dy/dx = 2x-4 \qquad 3x+2 = \left(\frac{3}{2}\right)(2x-4)+8$

$$I = \frac{3}{2}\int\left(\frac{2x-4}{x^2-4x+1}\right)dx + \frac{3}{2}\int\frac{8}{x^2-4x+1}dx$$

$$I = \frac{3}{2}\log_e\left(x^2-4x+1\right)+12\int\frac{dx}{x^2-4x+1}+K$$

$$x^2-4x+1=(x-2)^2-3 \qquad z = x-2 \quad a=\sqrt{3}$$

$$I_1 = \int\frac{dx}{x^2-4x+1} = \int\frac{dx}{z^2-a^2}$$

$$\frac{1}{z^2-a^2} = \left(\frac{1}{2a}\right)\left(\frac{1}{z-a}-\frac{1}{z+a}\right)$$

$$I_1 = \left(\frac{1}{2a}\right)\int\left(\left(\frac{1}{z-a}-\frac{1}{z+a}\right)\right)dx = \left(\frac{1}{2a}\right)\log_e\left(\frac{z-a}{z+a}\right)$$

$$I_1 = \left(\frac{1}{2\sqrt{3}}\right)\log_e\left(\frac{x-2-\sqrt{3}}{x-2+\sqrt{3}}\right)$$

$$I = \frac{3}{2}\log_e\left(x^2-4x+1\right)+2\sqrt{3}\log_e\left(\frac{x-2-\sqrt{3}}{x-2+\sqrt{3}}\right)+K$$

A26

$$I = \int_1^{\sqrt{3}} \frac{1+x}{x^2\left(1+x^2\right)} dx$$

$$\frac{1+x}{x^2\left(1+x^2\right)} = \frac{Ax+B}{x^2} + \frac{Cx+D}{\left(1+x^2\right)}$$

$$N = 1+x = \left(Ax+B\right)\left(1+x^2\right) + \left(Cx+D\right)\left(x^2\right)$$

$$N = \left(A+C\right)x^3 + \left(B+D\right)x^2 + Ax + B$$

$$A+C=0 \quad B+D=0 \quad A=1 \quad B=1 \quad \Rightarrow \quad C=-1 \quad D=-1$$

$$\frac{1+x}{x^2\left(1+x^2\right)} = \frac{1}{x} + \frac{1}{x^2} - \frac{x}{1+x^2} - \frac{1}{1+x^2}$$

$$I = \int_1^{\sqrt{3}} \left(\frac{1}{x} + \frac{1}{x^2} - \frac{x}{1+x^2} - \frac{1}{1+x^2} \right) dx$$

$$\int \frac{dx}{x} = \log_e\left(x\right) \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$I = \left[\log_e\left(x\right) - \frac{1}{x} - \tfrac{1}{2}\log_e\left(1+x^2\right) - \tan^{-1}\left(x\right) \right]_1^{\sqrt{3}}$$

$$I = \log_e\left(\sqrt{3}\right) - \frac{1}{\sqrt{3}} + 1 - \tfrac{1}{2}\log_e\left(2\right) - \tan^{-1}\left(\sqrt{3}\right) + \tan^{-1}\left(1\right)$$

$$\tan^{-1}\left(\sqrt{3}\right) = \pi/3 \quad \tan^{-1}\left(1\right) = \pi/4$$

$$I = 1 - \frac{1}{\sqrt{3}} + \log_e\left(\sqrt{\frac{3}{2}}\right) - \frac{\pi}{12}$$

A27

$$I=\int_0^1\left(e^x-1\right)^{1/2}dx$$

$$u^2=e^x-1 \qquad 2u\,du=e^x\,dx \qquad dx=\frac{2u}{1+u^2} \qquad x=0\rightarrow u=0 \qquad x=1\rightarrow u=\sqrt{e-1}$$

$$I=2\int_0^{\sqrt{e-1}}\frac{u^2}{1+u^2}du$$

$$\frac{u^2}{1+u^2}=A+\frac{B}{1+u^2}=\frac{A+u^2+B}{1+u^2} \qquad A=1 \quad B=-1 \qquad \frac{u^2}{1+u^2}=1-\frac{1}{1+u^2}$$

$$I=2\int_0^{\sqrt{e-1}}\left(1-\frac{1}{1+u^2}\right)du$$

$$\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$I=2\left[u-\tan^{-1}(u)\right]_0^{\sqrt{e-1}}$$

$$I=2\left(\sqrt{e-1}-\tan^{-1}\left(\sqrt{e-1}\right)\right)$$

A28

$$I=\int\frac{dx}{x^2\sqrt{1+x}}$$

$$u^2=1+x \quad 2u\,du=dx \quad x^2=\left(u^2-1\right)^2=\left(1-u^2\right)^2 \quad \sqrt{1+x}=u$$

$$I=2\int\frac{du}{\left(1-u^2\right)^2}$$

$$u=\cos\theta \quad du=-\sin\theta\,d\theta \quad 1-u^2=\sin^2\theta$$

$$I=-2\int\frac{d\theta}{\sin^3\theta}$$

$$t=\tan\left(\theta/2\right) \quad dt=\frac{1}{2}\left(1+\tan^2(\theta/2)\right)d\theta=\frac{1}{2}\left(1+t^2\right)d\theta \quad d\theta=\frac{2}{1+t^2}dt$$

$$\sin\theta=\frac{2t}{1+t^2} \quad \frac{1}{\sin^3\theta}=\frac{\left(1+t^2\right)^3}{8t^3}$$

$$I=-2\int\left(\frac{\left(1+t^2\right)^3}{8t^3}\right)\left(\frac{2}{1+t^2}\right)dt$$

$$I = -\frac{1}{2}\int\left(\frac{1+2t^2+t^4}{t^3}\right)dt = -\frac{1}{2}\int\left(t^{-3}+2t^{-1}+t\right)dt$$

$$I = \left(\frac{1}{4t^2} - \log_e(t) - \frac{1}{4}t^2\right) + K$$

$$\cos\theta = \frac{1-t^2}{1+t^2} \quad \cos\theta + \cos\theta t^2 = 1-t^2 \quad t^2 = \frac{1-\cos\theta}{1+\cos\theta}$$

$$I = \left(\frac{1+\cos\theta}{4(1-\cos\theta)} - \log_e\left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right) - \frac{1}{4}\left(\frac{1-\cos\theta}{1+\cos\theta}\right)\right) + K$$

$$u = \cos\theta$$

$$I = \left(\frac{1+u}{4(1-u)} - \frac{1}{4}\left(\frac{1-u}{1+u}\right) - \log_e\left(\sqrt{\frac{1-u}{1+u}}\right)\right) + K$$

$$I = \left(\frac{u}{(1-u^2)} - \frac{1}{2}\log_e\left(\frac{1-u}{1+u}\right)\right) + K$$

$$u = \sqrt{1+x}$$

$$I = -\frac{\sqrt{1+x}}{x} - \frac{1}{2}\log_e\left(\frac{1-\sqrt{1+x}}{1+\sqrt{1+x}}\right) + K$$

A29

$$I = \int \frac{9x-2}{2x^2-7x+3}\,dx$$

$$\frac{9x-2}{2x^2-7x+3} = \frac{9x-2}{(x-3)(2x-1)} = \frac{A}{x-3} + \frac{B}{2x-1}$$

$$2Ax - A + Bx - 3B = 9x - 2$$

$$A = 5 \quad B = -1$$

$$I = \int \left(\frac{5}{x-3} - \frac{1}{2\left(x-\frac{1}{2}\right)} \right) dx$$

$$I = 5\log_e(x-3) - \tfrac{1}{2}\log_e\left(x-\tfrac{1}{2}\right) + K$$

A30

$$I = \int \frac{3x^2 - 2x + 1}{\left(x^2+1\right)\left(x^2+2\right)} dx$$

$$\frac{3x^2 - 2x + 1}{\left(x^2+1\right)\left(x^2+2\right)} = \frac{A + Bx}{x^2+1} + \frac{C + Dx}{x^2+2}$$

$$3x^2 - 2x + 1 = Ax^2 + 2A + Bx^3 + 2Bx + Cx^2 + Dx^3 + C + Dx$$

$$3x^2 - 2x + 1 = (B + D)x^3 + (A + C)x^2 + (2B + D)x + 2A + C$$

$$D = -B \quad C = 3 - A \quad B = -2 \quad D = 2 \quad A = -2 \quad C = 5$$

$$I = \int \left(-2\left(\frac{1+x}{x^2+1} \right) + \frac{5+2x}{x^2+2} \right) dx$$

$$I = \int \left(-2\left(\frac{1}{x^2+1} + \frac{x}{x^2+1} \right) + \frac{5}{x^2+2} + \frac{2x}{x^2+2} \right) dx$$

$$I = \log_e\left(x^2+2\right) - \log_e\left(x^2+1\right) + \frac{5}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - 2\tan^{-1}\left(x\right) + K$$

A31

$$I = \int \frac{2x^2 + 3x - 1}{x^3 - x^2 + x - 1}\,dx$$

$$x^3 - x^2 + x - 1 = (x-1)(x^2+1)$$

$$\frac{2x^2 + 3x - 1}{x^3 - x^2 + x - 1} = \frac{A}{x-1} + \frac{B}{x^2+1}$$

$$2x^2 + 3x - 1 = Ax^2 + A + Bx - B$$

$$A = 2 \quad B = 3$$

$$I = \int \left(2\left(\frac{1}{x-1}\right) + 3\left(\frac{1}{x^2+1}\right)\right) dx$$

$$I = 2\log_e(x-1) + 3\tan^{-1}(x) + K$$