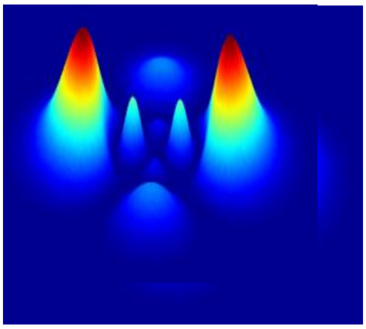

ADVANCED HIGH SCHOOL MATHEMATICS

CONICS

EXERCISES

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1

Consider the equation $4x^2 - 25y^2 = 100$

1A

What type of curve does the equation correspond too? Is the eccentricity e of the curve: $e = 1$; $e < 1$, $e > 1$; $e = 0$?

1B

Give the Cartesian coordinates for the vertices and focal points. Calculate the eccentricity e of the curve?

1C

State the equations for the directrices and asymptotes of the curve.

1D

The point P on the curve has the X-coordinate $x_P = 10$ and $y_P > 0$. Where does the tangent to the curve at the point P cut the X-axis and Y-axis? Where does the normal to the tangent at the point P intersect the X-axis and the Y-axis

1E

Sketch the curve showing the vertices, focal points, the asymptotes, directrices, and the points where the tangent and the normal intersect the X-axis and Y-axis. For your sketch: X-axis (-30 to +30) and Y-axis (-30 to +30).

Solution

1A

The equation $4x^2 - 25y^2 = 100$ corresponds to the curve of a **hyperbola**.

The **eccentricity** of hyperbolas is $e > 1$.

1B

A general expression for a hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The equation of the hyperbola $4x^2 - 25y^2 = 100$ can be re-written as

$$\frac{x^2}{5^2} - \frac{y^2}{2^2} = 1 \quad \Rightarrow \quad a = 5 \quad b = 2$$

The **vertices** of the hyperbola are $A_1(-a, 0)$ and $A_2(a, 0)$

$A_1(-5,0)$ and $A_2(5,0)$

The **focal length** c is $c^2 = a^2 + b^2 \quad c = \sqrt{a^2 + b^2} = \sqrt{25+4} = \sqrt{29} = 5.3852$

The **focal points** are $F_1(-c, 0)$ and $F_2(c, 0)$

$F_1(-\sqrt{29},0)$ and $F_2(\sqrt{29},0)$ or $F_1(-5.3852, 0)$ and $F_2(5.3852, 0)$

The **eccentricity** is $e = \frac{c}{a} = \frac{\sqrt{29}}{5} = 1.0770 > 1$

1C

The equations for the **directrices** are

$$x = \pm\frac{a^2}{c} \quad \Rightarrow \quad x = \frac{-25}{\sqrt{29}} = -4.6424 \qquad x = \frac{+25}{\sqrt{29}} = 4.6424$$

The equations for the **asymptotes** are

$$y = \pm\frac{b}{a}x \quad \Rightarrow \quad y = -\frac{2}{5}x = -0.4000\,x \qquad y = \frac{2}{5}x = 0.4000\,x$$

1D

The coordinates of the point P are (x_P, y_P)

$$x_P = 10 \quad y_P = \sqrt{4\left(\frac{100}{25}\right) - 1} = \sqrt{12} = 2\sqrt{3}$$

The equation of the straight line for the tangent is $y = M_1 x + B_1$

The gradient of the curve is given by the first derivative of the function

$$\left(\frac{2}{25}\right)x - \left(\frac{2}{4}\right)y\left(\frac{dy}{dx}\right) = 0 \qquad dy/dx = \left(\frac{4}{25}\right)\left(\frac{x}{y}\right)$$

The gradient M_1 at the point P

$$x_P = 10 \quad y_P = 2\sqrt{3} \quad M_1 = \left(\frac{4}{25}\right)\left(\frac{10}{2\sqrt{3}}\right) = \left(\frac{4}{5\sqrt{3}}\right) = 0.4619$$

The intercept B_1 of the tangent is

$$B_1 = y_P - M_1 x_P = 2\sqrt{3} - 10\left(\frac{4}{5\sqrt{3}}\right) = \frac{-2}{\sqrt{3}} = 1.1547$$

The tangent intersects the X-axis at the point T

$$y_T = 0 \quad x_T = -\frac{B_1}{M_1} = -\left(\frac{-2}{\sqrt{3}}\right)\left(\frac{5\sqrt{3}}{4}\right) = 2.5$$

The tangent intersects the Y-axis at the Point U

$$x_U = 0 \quad y_U = B_1 = \frac{-2}{\sqrt{3}} = -1.1547$$

The equation of the straight line for the normal is $y = M_2 x + B_2$

where $M_2 = \frac{-1}{M_1} = -\left(\frac{5\sqrt{3}}{4}\right) = 2.1651$

The intercept B_2 of the normal is

$$B_2 = y_P - M_2 x_P = 2\sqrt{3} + 10\left(\frac{5\sqrt{3}}{4}\right) = 14.5\sqrt{3} = 25.1147$$

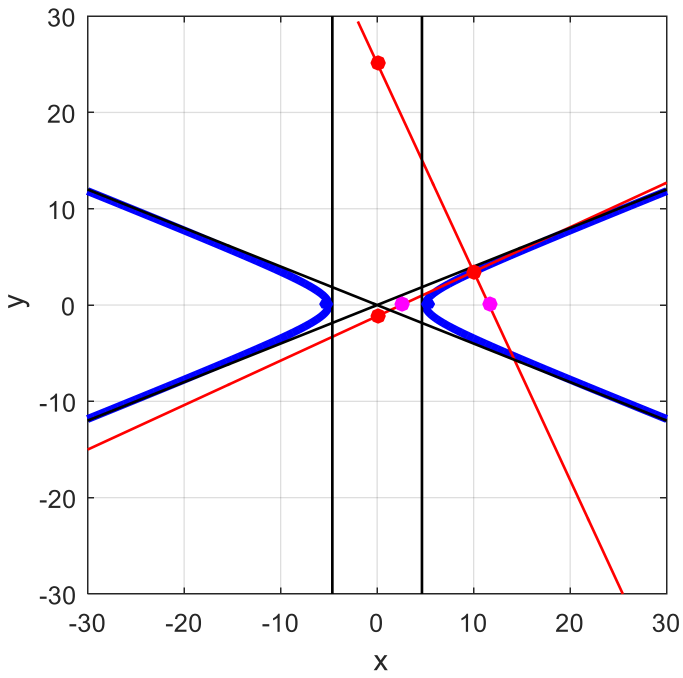
The normal intersects the X-axis at the point R

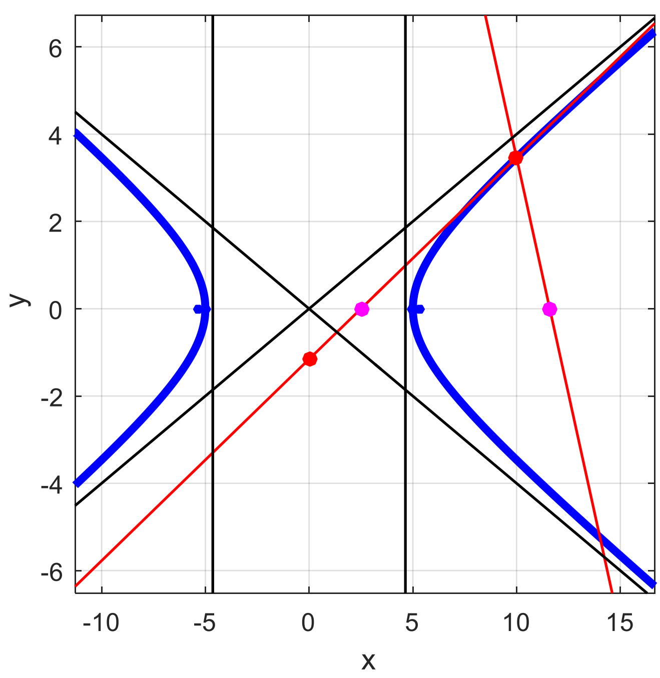
$$y_R = 0 \quad x_R = -\frac{B_2}{M_2} = -\left(14.5\sqrt{3}\right)\left(\frac{-4}{5\sqrt{3}}\right) = +11.600$$

The normal intersects the Y-axis at the Point S

$$x_S = 0 \quad y_S = B_2 = 14.5\sqrt{3} = 25.1147$$

1E





In the graphs, the vertices and focal points blue dots) are very close to each other

2

Consider the equation $xy = k^2$ k is a constant

The origin of the Cartesian coordinate system is O(0, 0). The two points P(x_P, y_P) and Q(x_Q, y_Q) lie on the curve and $x_P = k\,p$ and $x_Q = k\,q$.

2A

What type of curve does the equation correspond too?

How are the distances of a vertex a and focal point c from the origin related to the constant k? Show that the eccentricity e is $e = \sqrt{2}$.

2B

State the Cartesian coordinates for the vertices and focal points of the curve in terms of k.

2C

State the equations for the asymptotes, axes of symmetry, and the directrices of the curve.

2D

The tangent to the curve at the point P cuts the X-axis and Y-axis at the points T and U respectively.

Show that the equation of the tangent at the point P is $x + p^2 y = 2k\,p$.

Show that the points O, T, and U are on a circle with centre P.

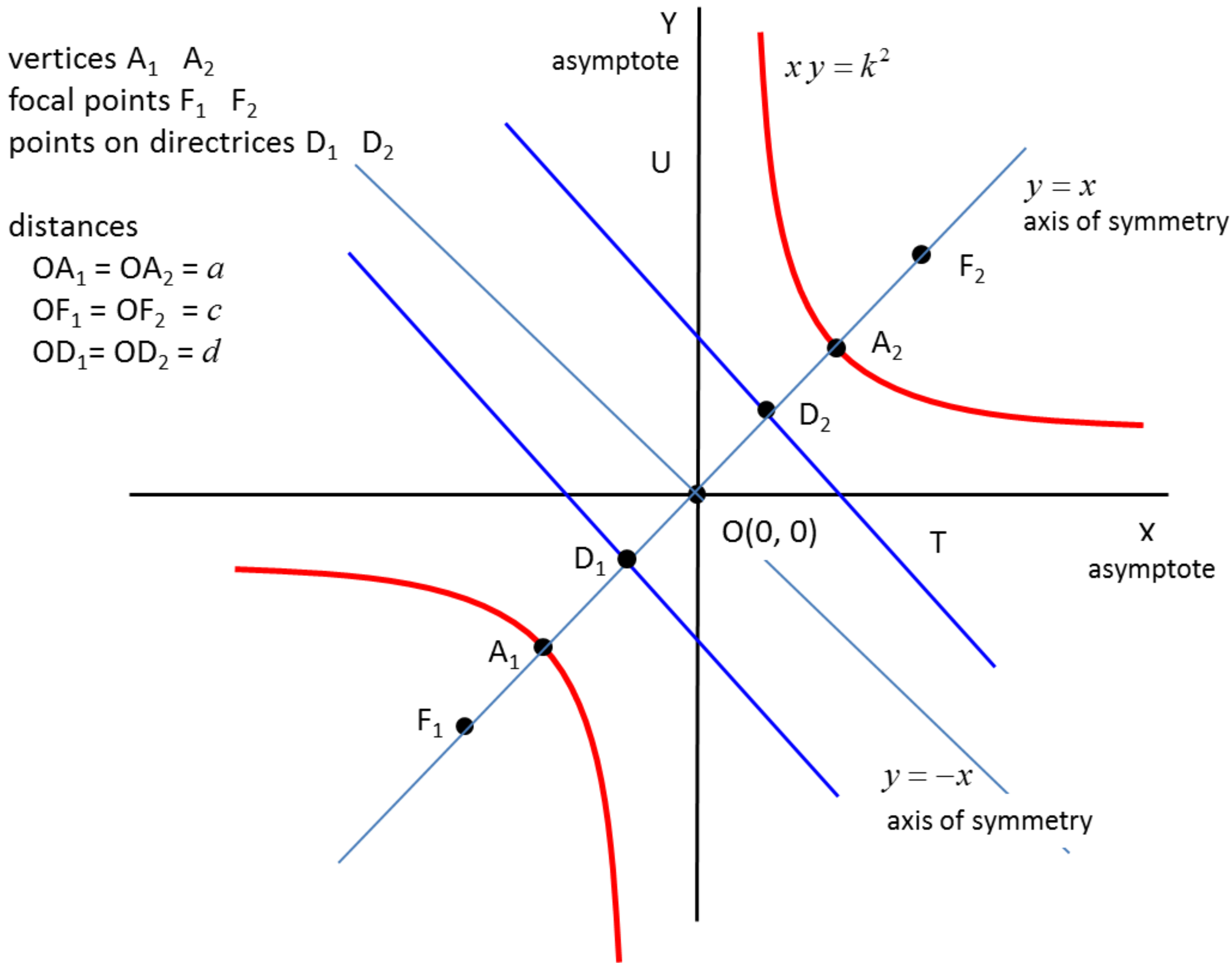
2E

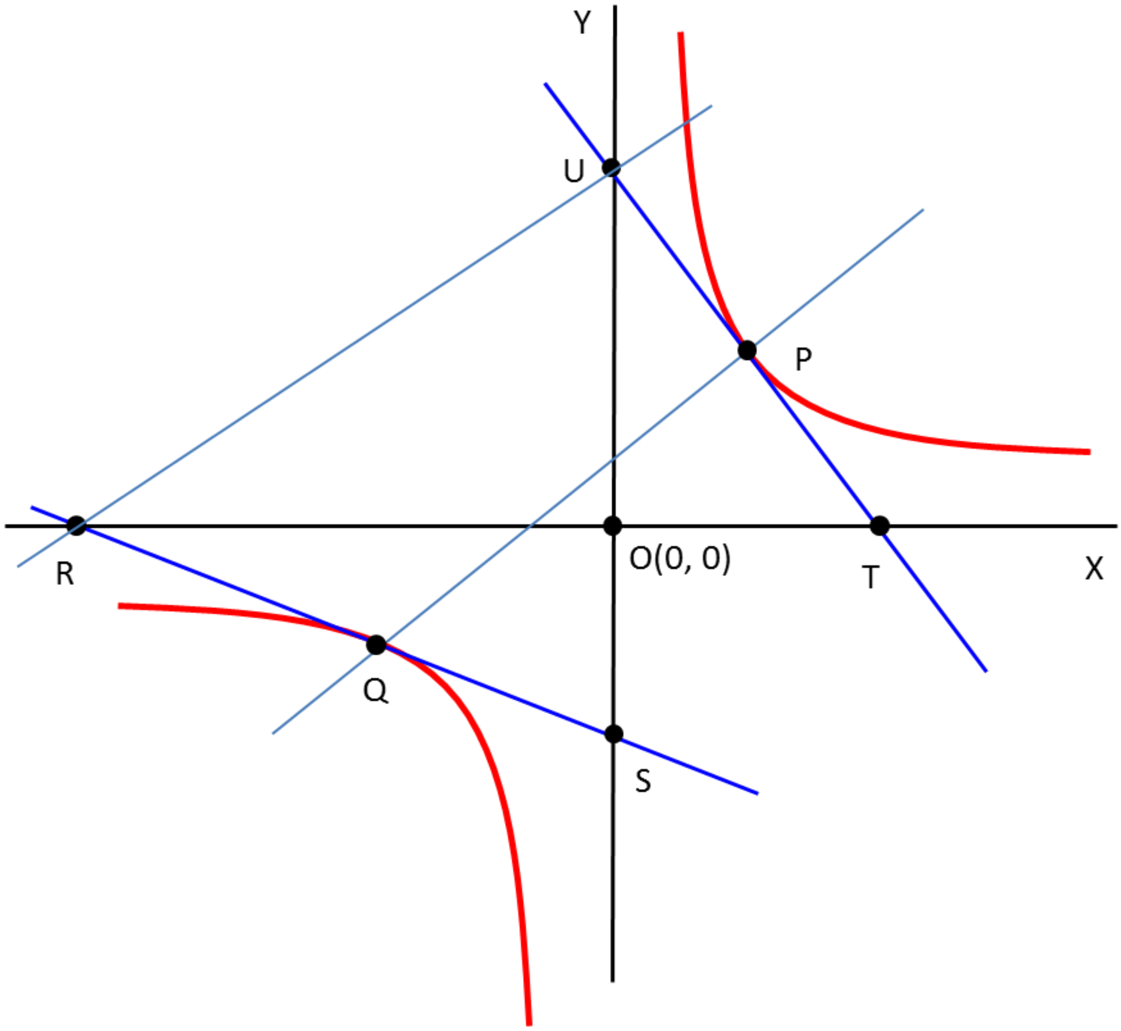
The tangent to the curve at the point Q cuts the X-axis and Y-axis at the points R and S respectively. Show that UR is parallel to PQ.

Solution

1A

Sketch diagrams of the curve and label key features before answering the questions





The equation for the curve of a **rectangular hyperbola** with the **openings in the first and third quadrants** is

$$x\,y = \frac{a^2}{2}$$ where a is the distance from a vertex to the origin.

Therefore the equation $x\,y = k^2$ is a rectangular hyperbola opening in the first and third quadrants where

$$k = \frac{a}{\sqrt{2}} \qquad a = \sqrt{2}\,k$$

The **focal length** c for the rectangular hyperbola is

$$a = b \quad c^2 = a^2 + b^2 = 2a^2 \quad c = \sqrt{2}\,a \quad a = \sqrt{2}\,k \quad c = 2k$$

The **eccentricity** of hyperbolas is given by

$$e = \frac{c}{a} = \frac{2k}{\sqrt{2}\,k} = \sqrt{2}$$

2B

The distances from the origin to the vertices A_1 and A_2 is a,

therefore **vertices** of the hyperbola are $A_1\left(-a/\sqrt{2}, -a/\sqrt{2}\right)$ and

$A_2\left(a/\sqrt{2}, a/\sqrt{2}\right)$

$$a = \sqrt{2}\,k \qquad k = a/\sqrt{2}$$

$A_1(-k, -k)$ and $A_2(k, k)$

Alternatively, the vertices are given by the intersection of the hyperbola $x\,y = k^2$ and the straight line x = y.

$$x = y \qquad x\,y = k^2 \qquad \Rightarrow \qquad x = \pm k \qquad y = \pm k$$

For a rectangular hyperbola a = b and the focal length c is

$$c^2 = a^2 + b^2 = 2\,a^2 \qquad c = \sqrt{2}\,a$$

The focal length is c where $c = \sqrt{2}\,a = 2k$, therefore, the coordinates of the focal points are

$F_1\left(-c/\sqrt{2}, -c/\sqrt{2}\right)$ or $F_1\left(-\sqrt{2}\,k, -\sqrt{2}\,k\right)$

$F_2\left(c/\sqrt{2}, c/\sqrt{2}\right)$ or $F_2\left(\sqrt{2}\,k, \sqrt{2}\,k\right)$

2C

The equations for the **asymptotes** are

First quadrant   +X axis and + Y axis

Third quadrant   -X axis and –Y axis

X-axis $y = 0$ Y-axis $x = 0$

The rectangular hyperbola has symmetrical openings in the first and third quadrants. Therefore, there are two axes of symmetry

The lines $y = x$ and $y = -x$

The directrices must be lines that are parallel to the axis of symmetry $y = -x$ and the distance d from this line to the axis of symmetry is

$$d = \frac{a^2}{c} \quad a = \sqrt{2}\,k \quad c = 2k \quad d = k$$

Let the equations for the **directrices** be of the form $y = -x \pm B$ with one directrix passing through the point $D_1\left(-k/\sqrt{2},\ -k/\sqrt{2}\right)$ and the other through D_2 $\left(k/\sqrt{2},\ k/\sqrt{2}\right)$, therefore , $B = \sqrt{2}\,k$. Hence, the equations for the two directrices are

$$y = -x + \sqrt{2}\,k \quad \text{and} \quad y = -x - \sqrt{2}\,k\ .$$

2D

The coordinates of the point P are (x_P, y_P)

$$x_P = k\,p \qquad y_P = \frac{k^2}{k\,p} = \frac{k}{p}$$

The equation of the straight line for the tangent is $y = M_1\,x + B_1$

The gradient of the curve is given by the first derivative of the function $x\,y = k^2$

$$y + x\left(\frac{dy}{dx}\right) = 0 \qquad dy/dx = -\left(\frac{y}{x}\right) = -\left(\frac{k^2}{x^2}\right)$$

The gradient M_1 at the point P

$$x_P = k\,p \qquad y_P = \frac{k}{p} \qquad M_1 = -\left(\frac{y_P}{x_P}\right) = -\frac{1}{p^2}$$

The intercept B_1 of the tangent is

$$B_1 = y_P - M_1\,x_P = \frac{k}{p} + \left(\frac{1}{p^2}\right)(k\,p) = \frac{2\,k}{p}$$

Hence, the equation of the tangent is

$$x + p^2\,y = 2\,k\,p$$

The tangent intersects the X-axis at the point T

$$y_T = 0 \qquad x_T = 2k\,p$$

The tangent intersects the Y-axis at the Point U

$$x_U = 0 \qquad y_U = \frac{2k}{p}$$

If the points O, T and U lie on a circle with centre P then the distance OP, TP and UP must be equal

$$d_{OP}{}^2 = x_P{}^2 + y_P{}^2 = k^2 p^2 + \frac{k^2}{p^2}$$

$$d_{TP}{}^2 = \left(x_P - x_T\right)^2 + y_P{}^2 = \left(k\,p - 2k\,p\right)^2 + \frac{k^2}{p^2} = k^2 p^2 + \frac{k^2}{p^2}$$

$$d_{UP}{}^2 = x_P{}^2 + \left(y_P - y_U\right)^2 = k^2 p^2 + \left(\frac{k}{p} - \frac{2k}{p}\right)^2 = k^2 p^2 + \frac{k^2}{p^2}$$

Therefore, all the points O, T and U lie on a circle with centre P

2E

From part (D), the equation of tangent and coordinates of the points R and S are

$$x + q^2 y = 2\,k\,q$$

The tangent intersects the X-axis at the point R

$$y_R = 0 \quad x_R = 2k\,q$$

The tangent intersects the Y-axis at the Point S

$$x_S = 0 \quad y_S = \frac{2k}{q}$$

The slope of the line PQ is

$$m_{PQ} = \frac{k\,/\,p - k\,/\,q}{k\,p - k\,q} = \frac{-1}{p\,q}$$

The slope of the line UR is

$$m_{UR} = \frac{2k\,/\,p - 0}{0 - 2k\,q} = \frac{-1}{p\,q}$$

The slopes are equal, hence the two lines are parallel.