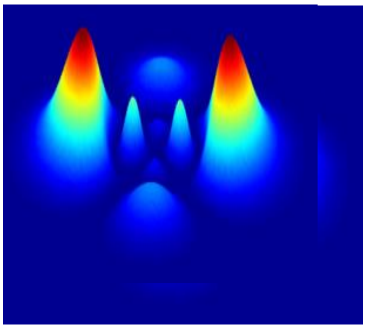

ADVANCED HIGH SCHOOL MATHEMATICS

COMPLEX NUMBERS

EXERCISES

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To find a Question or Answer use the find function: *control f*

Question 4   Q4

Answer 15   A15

QUESTIONS

Q1

Consider the two complex variables

$$z_1 = 2 - \sqrt{3}\, i \quad \text{and} \quad z_2 = 1 + \sqrt{3}\, i$$

Find the following

The complex conjugates of z_1 and z_2
The magnitudes (moduli) of z_1 and z_2
The arguments of z_1 and z_2
The polar and exponential forms of z_1 and z_2

$z_1 + z_2 \qquad z_1 - \overline{z}_2 \qquad z_1 + \sqrt{3}i\ z_2$

$z_1\, z_2$ magnitude and argument of $z_1\, z_2$

$\dfrac{z_1}{z_2}$ magnitude and argument of $\dfrac{z_1}{z_2}$

z_1^{-12} and z_2^{24} in its simplest form

Q2

Sketch the region on the Argand diagram defined by the relationship

$$z^2 + \overline{z}^2 \le 8$$

Q3

Sketch the region on the Argand diagram defined by the inequalities

$$|z - i| \le 1 \qquad |z + 2| \ge 2$$

Q4

Sketch the region on the Argand diagram defined by the inequality

$$\left|\frac{1}{z} + i\right| \le 1$$

Q5

Let z and w be two complex numbers

$$z = 2 - 2i \quad \text{and} \quad w = 1 + \sqrt{3}\,i$$

Check that all the answers are correct.

Complex number *z*

$$z = x + i\,y = \cos\theta + i\sin\theta = |z|\exp(i\,\theta)$$

Magnitude (modulus or absolute value) $|z| = \sqrt{x^2 + y^2}$

Argument $\theta = \arg(z) = \operatorname{atan}\left(\frac{y}{x}\right) \qquad \operatorname{atan} \equiv \tan^{-1}$

Complex conjugate $\bar{z} = x - i\,y$

A(1) $|z|$ 2.8284

A(2) θ_z -45^o

A(3) $|w|$ 2.0000

A(4) θ_w 60^o

A(5) $|\bar{z}|$ 2.8284

A(6) $|\bar{w}|$ 2.0000

A(7) $w - z$ -1.0000 + 3.7321i

A(8) $2z + i\,w$ 2.2679 - 3.0000i

A(9) $\overline{1 - z}$ -1.0000 - 2.0000i

A(10) $z\,w$ 5.4641 + 1.4641i

A(11) $z\,\bar{w}$ -1.4641 - 5.4641i

A(12) z^4 -64

A(13) w^{10} -5.1200e+02 - 8.8681e+02i

A(14) $\frac{z}{\bar{z}}$ - 1.0000i

A(15) $\frac{i\,z}{w}$ 1.3660 - 0.3660i

A(16) $\frac{4}{\bar{w}}$ 1.0000 + 1.7321i

Q6

Specify the real and imaginary parts of the complex number z and the complex conjugate of z

$$z = 55 - 22\,i$$

Q7

Plot the complex number $z = -3 + 2\,i$ on an Argand diagram (complex plane) and determines its modulus and argument. Do the same for the complex conjugate of z.

Q8

Convert the complex number $z = 3 - 4\,i$ to polar form and exponential form. Give the polar and exponential forms for the complex conjugate of z.

Q9

Graph the complex numbers $z_1 = i$ and $z_2 = -i$ on Argand diagram. State the polar and exponential forms of these complex numbers.

Q10

Find the rectangular, polar and exponentials form of the complex number

$$z = 6\angle\left(\frac{\pi}{3}\ \text{rad}\right)$$

Q11

Verify each of the following relationships

(a) $\frac{1}{2}(1+i)^2 = i$ (b) $\sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$ (c) $\sqrt{i} = e^{i\left(\frac{\pi}{4}\right)}$

(d) $\sqrt{-i} = \frac{1}{\sqrt{2}}(1-i)$ (e) $\sqrt{-i} = e^{i\left(-\frac{\pi}{4}\right)}$

Q12

Rationalize the complex numbers

$$z_1 = \frac{2}{2+i} \quad z_2 = \frac{5\,i}{1-2\,i}$$

Q13

Find the simplest rectangular form of

$$z_1 = (1-i)^4 \quad z_2 = \left(\sqrt{2}-i\right) - i\left(1-i\,\sqrt{2}\right) \quad z_3 = \frac{10}{(1-i)(2-i)(3-i)}$$

Q14

If $z = -1+i$ show that $z^7 = -8\,(1+i)$

Q15

Prove the following relationships

$$\cos^4(\theta) = \left(\frac{1}{8}\right)\left[\cos(4\theta) + 4\cos(2\theta) + 3\right]$$

$$\sin^4(\theta) = \left(\frac{1}{8}\right)\left[\cos(4\theta) - 4\cos(2\theta) + 3\right]$$

Q16

Show that

$$\frac{\left(\sqrt{3}+i\right)^6\left(1+i\sqrt{3}\right)^4}{\left(\sqrt{3}-i\right)^4\left(1-i\sqrt{3}\right)^3} = 8$$

Q17

Find the cubic root of 2. Plot the roots on an Argand diagram.

Q18

Find the fourth roots of $3+2i$. Plot the roots on an Argand diagram.

Q19

Prove the following

$$\cos\theta = \frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right) \qquad \sin\theta = \frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right) \qquad \tan\theta = -i\left(\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}\right)$$

Q20

Using the results of exercise (19) show

$$\tan(\theta/2) = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$$

$$\sin(\theta/2) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos(\theta/2) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Q21

Solve the equation $z^5 - 1 = 0$

Q22

Show the region on an Argand diagram that satisfy the conditions

$$\mathrm{Re}(z) \le 3 \qquad \mathrm{Re}(z) > -3$$

$$\mathrm{Im}(z) < 2 \qquad \mathrm{Re}(z) > -3$$

Q23

Show on an Argand diagram the complex numbers z that satisfy the condition

$$|z - z_1| = |z - z_2| \qquad z_1 = 1 + i \qquad z_2 = -3 + i$$

Q24

If $z_1 = 1 + i$ then show on an Argand diagram

$$z - z_1 = 2\left(\cos\left(\pi/4\right) + i\sin\left(\pi/4\right)\right)$$

Q25

If $z_1 = -1 + i$ then show on an Argand diagram

$$\left|z - z_1\right| = 2$$

Q26

If $z_1 = -1 + i$ then show on an Argand diagram

$$\theta = Arg\left(z - z_1\right) = \pi/4$$

Q27

Show that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad \arg\left(\frac{z_1}{z_2}\right) = \arg\left(z_1\right) - \arg\left(z_2\right)$$

$$Arg\left(z - z_1\right) - Arg\left(z - z_2\right) = Arg\left(\frac{z - z_1}{z - z_2}\right)$$

Sketch and comment on the locus of

$$Arg\left(z - z_1\right) - Arg\left(z - z_2\right) = Arg\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{6}$$

where

$$z_1 = 1 + i \qquad z_2 = -2 + i$$

Q28

Sketch the region on an Argand diagram for the expression

$$2\left(z+\overline{z}\right)-z\,\overline{z}>8$$

Q29

Sketch the allowed region defined by the relationships

$$-\frac{\pi}{4}<Arg\left(z\right)<\frac{\pi}{4} \qquad \left|z-i\right|<3$$

ANSWERS

A1

$$\bar{z}_1 = 2 + \sqrt{3}i \qquad \bar{z}_2 = 1 - \sqrt{3}\,i$$

$$|z_1| = R_1 = \sqrt{4+3} = \sqrt{7} = 2.6458 \qquad |z_2| = R_2 = \sqrt{1+3} = 2.0000$$

$$\arg(z_1) = \theta_1 = \tan^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -0.7137 \text{ rad}$$

$$\arg(z_2) = \theta_2 = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 1.0472 \text{ rad} = \frac{\pi}{3} \text{ rad}$$

Polar form $\quad z_1 = R_1\left(\cos\theta_1 + i\sin\theta_1\right) \qquad z_2 = R_2\left(\cos\theta_2 + i\sin\theta_2\right)$

Exponential form $\quad z_1 = R_1\, e^{i\theta_1} \qquad z_2 = R_2\, e^{i\theta_2}$

$$R_1 = \sqrt{7} = 2.6458 \quad \theta_1 = -0.7137 \text{ rad}$$

$$R_2 = 2.0000 \quad \theta_2 = 1.0472 \text{ rad} = \frac{\pi}{3} \text{ rad}$$

$$z_1 + z_2 = 3 \qquad z_1 - \bar{z}_2 = 1$$

$$z_1 = 2 - \sqrt{3}\,i \qquad z_2 = 1 + \sqrt{3}\,i \quad \sqrt{3}\,i\,z_2 = -3 + \sqrt{3}\,i \quad z_1 + \sqrt{3}\,i\,z_2 = -1$$

$$z_1 z_2 = \left(2 - \sqrt{3}\,i\right)\left(1 + \sqrt{3}\,i\right) = 5 + \sqrt{3}\,i$$

$$|z_1 z_2| = \sqrt{25+3} = 5.2915 \qquad |z_1 z_2| = R_1 R_2 = (2.6458)(2.0000) = 5.2915$$

$$\arg(z_1 z_2) = \tan^{-1}\left(\frac{\sqrt{3}}{5}\right) = 0.3335 \text{ rad}$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = -0.7137 + 1.0472 = 0.3335$$

$$z_1 / z_2 = \frac{2-\sqrt{3}i}{1+\sqrt{3}\,i} = \frac{2-\sqrt{3}i}{1+\sqrt{3}\,i}\;\frac{1-\sqrt{3}\,i}{1-\sqrt{3}\,i} = -0.2500 - 1.2990i$$

$$\left|z_1 / z_2\right| = \sqrt{0.2500^2 + 1.2990^2} = 1.329$$

$$\left|z_1 / z_2\right| = R_1 / R_2 = \left(2.6458\right) / \left(2.0000\right) = 1.3229$$

$$\arg(z_1 / z_2) = \tan^{-1}\left(\frac{-1.2990}{0.2500}\right) = -1.7609 \text{ rad}$$

$$\arg(z_1 / z_2) = \arg(z_1) - \arg(z_2) = -0.7137 - 1.0472 = -1.7609$$

NB in the Argand diagram the complex number $\frac{z_1}{z_2}$ is in the third quadrant

$$z_1 = R_1 e^{i\theta_1} = 7^{1/2} e^{i(-0.7137)}$$

$$z_1^{-12} = 7^{(1/2)(-12)} e^{i(-0.7137)(-12)} = 7^{-6} e^{i(8.5644)} = 7^{-6} e^{i(8.5644 - 2\pi)} = 7^{-6} e^{i(2.2812)}$$

$$z_2 = R_2 e^{i\theta_2} = 2e^{i(\pi/3)}$$

$$z_2^{24} = 2^{24} e^{i(24\pi/3)} = 2^{24} e^{i(8\pi)} = 2^{24}$$

A2

$$z = x + i\,y \quad z^2 = x^2 - y^2 + i(2xy) \qquad \overline{z} = x - i\,y \quad \overline{z}^2 = x^2 - y^2 - i(2xy)$$

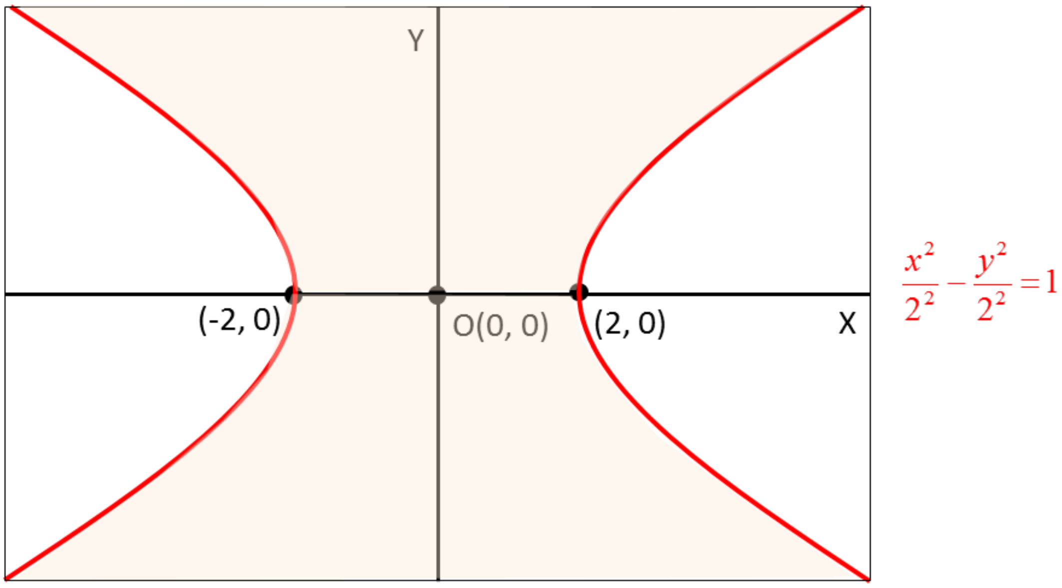
$$z^2 + \overline{z}^2 = 2\left(x^2 - y^2\right) \le 8 \qquad \frac{x^2}{2^2} - \frac{y^2}{2^2} \le 1$$

The equation $\frac{x^2}{2^2} - \frac{y^2}{2^2} = 1$ corresponds to a rectangular hyperbola with vertices at (-2, 0) and (2, 0).

$$x = 0 \quad y = 0 \quad \Rightarrow z^2 + \overline{z}^2 \le 8$$

$$x > 2 \quad y = 0 \quad \Rightarrow z^2 + \overline{z}^2 > 8$$

The shaded area is the defined region.



A3

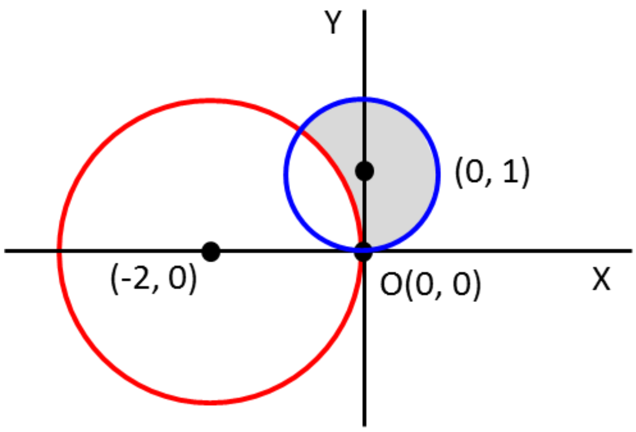
$|z-i|\le 1 \qquad |x+i(y-1)|\le 1 \qquad x^2+(y-1)^2\le 1$

Region corresponds to the area inside a circles of radius 1 and centre (0, 1)

$|z+2|\ge 2 \qquad |(x+2)+i\,y|\ge 2 \qquad (x+2)^2+y^2\ge 2$

Region corresponds to the area outside a circles of radius 2 and centre (-2, 0)

The shaded area is the region defined by $|z-i|\le 1 \qquad |z+2|\ge 2$

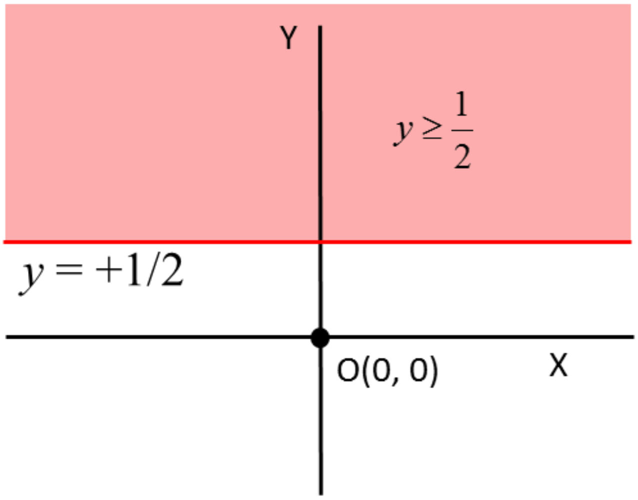


A4

$$\left|\frac{1}{z}+i\right|\le 1$$

$$\left|\frac{1}{z}+i\right| = \left|\frac{1+iz}{z}\right| = \frac{|1+iz|}{|z|}\le 1$$

$$|1+iz|\le |z|$$



$$z = x+iy$$
$$iz = -y+ix$$
$$1+iz = 1-y+ix$$
$$|z|^2 = x^2+y^2$$
$$|1+iz|^2 = (1-y)^2+x^2 = 1-2y+y^2+x^2$$

$$1-2y+y^2+x^2\le x^2+y^2$$
$$y\ge\frac{1}{2}$$

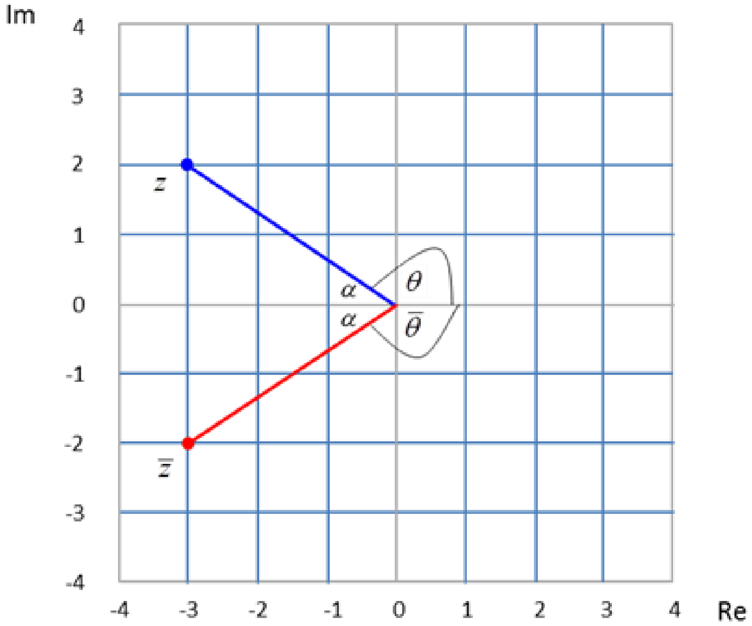
Region shown as shaded area

A6

$\mathrm{Re}(z) = 55 \quad \mathrm{Im}(z) = -22 \qquad \mathrm{Re}(\bar{z}) = 55 \quad \mathrm{Im}(\bar{z}) = 22$

A7

$\bar{z}$ is a reflection of z about the Re axis



$$z = x + i\ y \qquad \bar{z} = x - i\ y$$

$$|z| = |\bar{z}| = \sqrt{(-3)^2 + 2^2} = 3.6056$$

$$\tan\alpha = \frac{2}{3} \qquad \alpha = 0.5580 \text{ rad}$$

$$\theta = Arg(z) = (\pi - \alpha) = 2.5536 \text{ rad} = 146^{\circ}$$

$$\bar{\theta} = Arg(\bar{z}) = (-\pi + \alpha) = -2.5536 \text{ rad} = -146^{\circ}$$

A8

$$z = x + i\,y \quad \bar{z} = x - i\,y \quad z = R\left(\cos\theta + i\,\sin\theta\right) = R\,e^{\,i\,\theta}$$

$$|z| = |\bar{z}| = \sqrt{3^2 + 4^2} = 5$$

$$\tan\alpha = \frac{4}{3} \quad \alpha = 0.9273\,\text{rad}$$

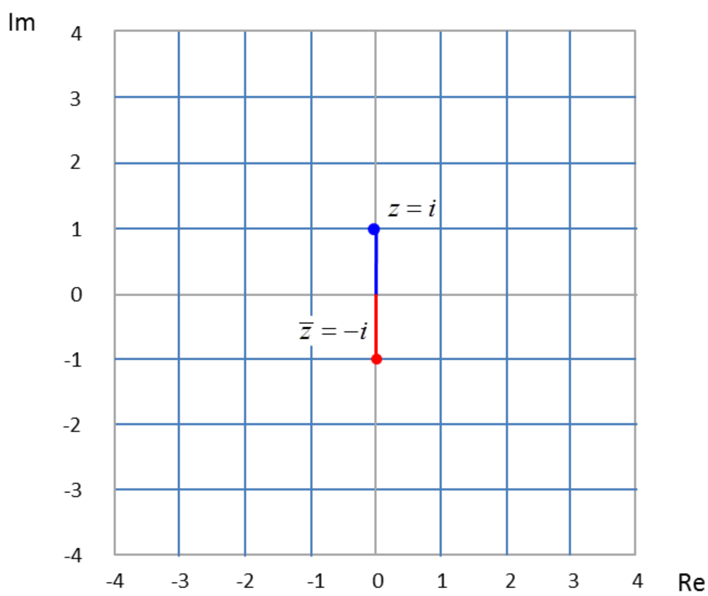
$$\theta = Arg(z) = -\alpha = -0.9273\,\text{rad} = -53^{o}$$

$$\bar{\theta} = Arg(\bar{z}) = \alpha = 0.9273\,\text{rad} = 53^{o}$$

$$z = 5\left[\cos\left(0.9273\right) - i\,\sin\left(0.9273\right)\right] = 5\,e^{\,i\,(-0.9273)}$$

$$\bar{z} = 5\left[\cos\left(0.9273\right) + i\,\sin\left(0.9273\right)\right] = 5\,e^{\,i\,(0.9273)}$$

A9



$$\theta_1 = \frac{\pi}{2} \quad z_1 = \cos\left(\pi/2\right) + i\,\sin\left(\pi/2\right) = e^{\,i\,\pi/2} = i$$

$$\theta_2 = -\frac{\pi}{2} \quad z_2 = \cos\left(-\pi/2\right) + i\,\sin\left(-\pi/2\right) = e^{\,i\,(-\pi/2)} = -i$$

A10

$$R = 6 \quad \theta = \pi / 3$$

$$x = R\cos\theta = 6\cos(\pi/3) = 3 \qquad y = R\sin\theta = 6\sin(\pi/3) = 5.1962$$

$$z = 3 + i\,(5.1962)$$

$$z = 6\left[\cos(\pi/3) + i\,\sin(\pi/3)\right] = 6\ e^{\,i\,(\pi/3)}$$

A11

$$\frac{1}{2}(1+i)^2 = \frac{1}{2}(1+2\,i-1) = i$$

$$i = e^{\,i(\pi/2)} \quad \sqrt{i} = \left[e^{\,i(\pi/2)}\right]^{1/2} = e^{\,i(\pi/4)} = \cos(\pi/4) + i\,\sin(\pi/4) = \left(\frac{1}{\sqrt{2}}\right)(1+i)$$

$$-i = e^{\,i(-\pi/2)} \quad \sqrt{-i} = \left[e^{\,i(-\pi/2)}\right]^{1/2} = e^{\,i(\pi/4)} = \cos(-\pi/4) + i\,\sin(-\pi/4) = \left(\frac{1}{\sqrt{2}}\right)(1-i)$$

A12

$$z_1 = \frac{2}{2+i} = \left(\frac{2}{2+i}\right)\left(\frac{2-i}{2-i}\right) = \frac{4-2\,i}{5} = \left(\frac{4}{5}\right) - \left(\frac{2}{5}\right)i$$

$$z_2 = \frac{5\,i}{1-2\,i} = \left(\frac{5\,i}{1-2\,i}\right)\left(\frac{1+2\,i}{1+2\,i}\right) = \frac{5\,i-10}{5} = -2+i$$

A13

$$z_1 = (1-i)^4$$
$$(1-i)^2 = 1-i-i-1 = -2\,i$$
$$(1-i)^4 = (-2\,i)^2 = -4$$
$$z_1 = (1-i)^4 = -4$$

$$z_2 = \left(\sqrt{2}-i\right) - i\left(1-i\sqrt{2}\right) = \sqrt{2}-i-i-\sqrt{2} = -2\,i$$

$$z_3 = \frac{10}{(1-i)(2-i)(3-i)}$$
$$= \frac{10(1+i)(2+i)(3+i)}{(1-i)(2-i)(3-i)(1+i)(2+i)(3+i)} = \frac{(10)(10\,i)}{(2)(5)(10)} = i$$

A14

$$R = \sqrt{2} \quad \tan\left(|y/x|\right) = \tan(1) = \pi/4 \quad \theta = \pi - \pi/4 = 3\pi/4$$

$$z = 2^{1/2}\, e^{\,i\,(3\pi/4)} \qquad z^7 = 2^{7/2}\, e^{\,i\,(3\pi/4)7} = 8\,e^{\,i\,(21\pi/4)} = 8\,e^{\,i\left(\frac{16+5}{4}\pi\right)} = 8\,e^{\,i\left(\frac{5}{4}\pi\right)}$$

$$x = 8\cos(5\pi/4) = -8 \quad y = 8\sin(5\pi/4) = -8$$

$$z^7 = -8(1+i)$$

A15

$$\left[\cos(\theta)+i\sin(\theta)\right]^2 = e^{i(2\theta)} = \cos(2\theta)+i\sin(2\theta)$$

$$\left[\cos(\theta)+i\sin(\theta)\right]^2 = \left[\cos^2(\theta)-\sin^2(\theta)\right]+i\left[2\cos(\theta)\sin(\theta)\right]$$

Equating real and imaginary parts

$$\cos(2\theta) = \cos^2(\theta)-\sin^2(\theta)$$

$$\sin(2\theta) = 2\cos(\theta)\sin(\theta)$$

Binomial Theorem: a very useful formula to remember for the expansion a function of the form $(a + b)^n$ where $n = 1, 2, 3, \ldots$

$$(a+b)^n = a^n + \frac{n}{1!}a^{(n-1)}b^1 + \frac{n(n-1)}{2!}a^{(n-2)}b^2 + \frac{n(n-1)(n-2)}{3!}a^{(n-3)}b^3 + \ldots$$

$$(a+b)^4 = a^4 + 4a^3b^1 + \frac{(4)(3)}{(2)(1)}a^2b^2 + \frac{(4)(3)(2)}{(3)(2)(1)}ab^3 + \frac{(4)(3)(2)(1)}{(4)(3)(2)(1)}b^4$$

$$(a+b)^4 = a^4 + 4a^3b^1 + 6a^2b^2 + 4ab^3 + b^4$$

$$\left[\cos(\theta)+i\sin(\theta)\right]^4 = e^{i(4\theta)} = \cos(4\theta)+i\sin(4\theta)$$

$$\left[\cos(\theta)+i\sin(\theta)\right]^4$$

$$= \cos^4(\theta)+(i)(4)\cos^3(\theta)\sin(\theta)+(i)^2(6)\cos^2(\theta)\sin^2(\theta)+(i)^3(4)\cos(\theta)\sin^3(\theta)+(i)^4\sin^4(\theta)$$

$$= \left[\cos^4(\theta)+\sin^4(\theta)-(6)\cos^2(\theta)\sin^2(\theta)\right]+i\left[(4)\cos^3(\theta)\sin(\theta)-(4)\cos(\theta)\sin^3(\theta)\right]$$

Equating real and imaginary parts

$$\cos(4\theta) = \cos^4(\theta)+\sin^4(\theta)-(6)\cos^2(\theta)\sin^2(\theta)$$

$$\sin(4\theta) = (4)\cos^3(\theta)\sin(\theta)-(4)\cos(\theta)\sin^3(\theta)$$

$$\cos(4\theta)+4\cos(2\theta)+3$$

$$= \quad \cos^4(\theta)+\sin^4(\theta)-(6)\cos^2(\theta)\sin^2(\theta)$$

$$+4\left[\cos^2(\theta)-\sin^2(\theta)\right)+3$$

$$=\cos^4(\theta)+\left[1-\cos(\theta)\right]^2-6\cos^2(\theta)\left[1-\cos^2(\theta)\right]$$

$$+8\cos^2(\theta)-1$$

$$=8\cos^4(\theta)$$

$$\cos^4(\theta)=\left(\frac{1}{8}\right)\left[\cos(4\theta)+4\cos(2\theta)+3\right]$$

$$\cos^4(\theta)=\left[1-\sin^2(\theta)\right]^2=1-2\sin^2(\theta)+\sin^4(\theta)$$

$$=\cos^2(\theta)+\sin^2(\theta)-2\sin^2(\theta)+\sin^4(\theta)$$

$$=\cos^2(\theta)-\sin^2(\theta)+\sin^4(\theta)$$

$$=\cos(2\theta)+\sin^4(\theta)$$

$$\sin^4(\theta)=\left(\frac{1}{8}\right)\left[\cos(4\theta)-4\cos(2\theta)+3\right]$$

A16

$$\frac{\left(\sqrt{3}+i\right)^6\left(1+i\sqrt{3}\right)^4}{\left(\sqrt{3}-i\right)^4\left(1-i\sqrt{3}\right)^3}=8$$

$$z_1=\sqrt{3}+i=2\,e^{i\pi/6}$$

$$z_1{}^6=2^6\,e^{i\pi}$$

$$z_2=1+i\sqrt{3}=2\,e^{i\pi/3}$$

$$z_2{}^4=2^4\,e^{i4\pi/3}$$

$$z_3=\sqrt{3}-i=2\,e^{-i\pi/6}$$

$$z_3{}^{-4}=2^{-4}\,e^{i2\pi/3}$$

$$z_4=1-\sqrt{3}i=2\,e^{-i\pi/3}$$

$$z_3{}^{-3}=2^{-3}\,e^{i\pi}$$

$$z_1\,z_2\,z_3\,z_4=2^{(6+4-4-3)}\,e^{i(\pi+4\pi/3+2\pi/3+\pi)}=2^3\,e^{i(\pi+4\pi/3+2\pi/3+\pi)}=8$$

A17

$$\sqrt[3]{2R} = ?$$

$$z = 2\left[\cos(2\pi) + i\sin(2\pi)\right]$$

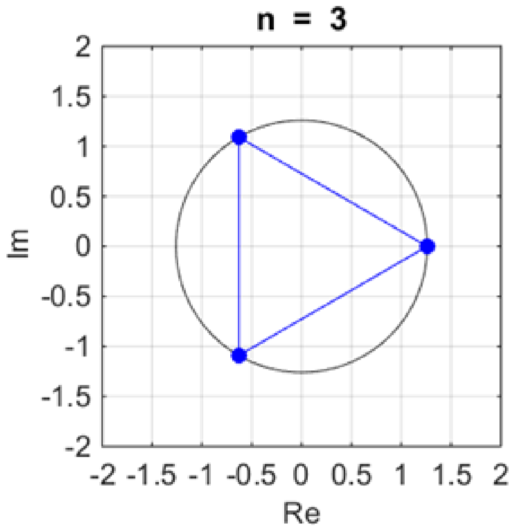
$$= 2\left[\cos(2\pi k) + i\sin(2\pi k)\right] \quad k = 0, 1, 2$$

$$w_k = z^{1/3} = 2^{1/3}\left[\cos(2\pi k/3) + i\sin(2\pi k/3)\right]$$

$$w_0 = 2^{1/3}\left[\cos(0) + i\sin(0)\right] = 2^{1/3}$$

$$w_1 = 2^{1/3}\left[\cos(2\pi/3) + i\sin(2\pi/3)\right] = 2^{1/3}\left[-1/2 + i\left(\sqrt{3}/2\right)\right]$$

$$w_2 = 2^{1/3}\left[\cos(4\pi/3) + i\sin(4\pi/3)\right] = 2^{1/3}\left[-1/2 - i\left(\sqrt{3}/2\right)\right]$$



A18

There are **four** roots.

$$\sqrt[4]{3+2\,i}=?$$

$$z=3+2\,i \qquad |z|=\sqrt{3^2+2^2}=13^{1/2} \qquad \theta=\arg(z)=\text{atan}(2/3)=0.5880\text{ rad}$$

$$z=13^{1/2}\left[\cos(\theta)+i\sin(\theta)\right]=13^{1/2}\,e^{i\theta}=13^{1/2}\,e^{i(\theta+2\pi k)} \qquad k=0,1,2,3$$

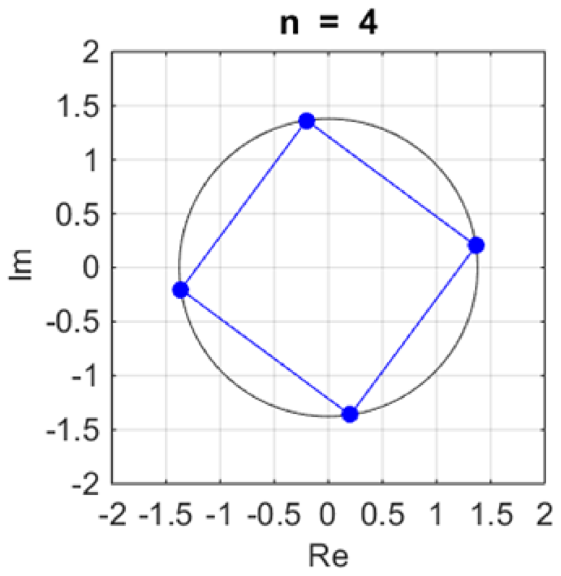
$$w_k=z^{1/4}=13^{1/8}\,e^{i(\theta/4+\pi k/2)} \qquad \theta/4=0.1651\text{ rad}$$

$$w_0=13^{1/8}\left[\cos(\theta/4)+i\sin(\theta/4)\right]=13^{1/8}\left[0.9892+i\,(0.1465)\right]$$

$$w_1=13^{1/8}\left[\cos(\theta/4+\pi/2)+i\sin(\theta/4+\pi/2)\right]=13^{1/8}\left[-0.1465+i\,(0.9892)\right]$$

$$w_2=13^{1/8}\left[\cos(\theta/4+\pi)+i\sin(\theta/4+\pi)\right]=13^{1/8}\left[-0.9892+i\,(-0.1465)\right]$$

$$w_3=13^{1/8}\left[\cos(\theta/4+3\pi/2)+i\sin(\theta/4+3\pi/2)\right]=13^{1/8}\left[0.1465+i\,(-0.9892)\right]$$



A19

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \qquad \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta \qquad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = -i\left(\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}\right)$$

A20

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = -i\left(\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}\right)$$

Replace θ by $\theta/2$

$$\tan(\theta/2) = -i\left(\frac{e^{i(\theta/2)} - e^{-i(\theta/2)}}{e^{i(\theta/2)} + e^{-i(\theta/2)}}\right) = -i\left(\frac{e^{i(\theta/2)} - e^{-i(\theta/2)}}{e^{i(\theta/2)} + e^{-i(\theta/2)}}\right)\left(\frac{e^{i(\theta/2)}}{e^{i(\theta/2)}}\right)$$

$$= -i\left(\frac{e^{i\theta} - 1}{e^{i\theta} + 1}\right) = -i\,\frac{z_1}{z_2}$$

$$z_1 = e^{i\theta} - 1 = (\cos\theta - 1) + i\sin\theta$$

$$z_2 = e^{i\theta} + 1 = (\cos\theta + 1) + i\sin\theta \quad \overline{z}_2 = e^{i\theta} + 1 = (\cos\theta + 1) - i\sin\theta$$

$$z = \frac{z_1}{z_2} = \frac{z_1\,\overline{z}_2}{z_2\,\overline{z}_2}$$

$$z_2\,\overline{z}_2 = (\cos\theta + 1)^2 + \sin^2\theta = \cos^2\theta + 2\cos\theta + \sin^2\theta = 2(1 + \cos\theta)$$

$$z = \frac{\left((\cos\theta - 1) + i\sin\theta\right)\left((\cos\theta + 1) - i\sin\theta\right)}{2(1 + \cos\theta)}$$

$$= \frac{(\cos\theta - 1)(\cos\theta + 1) - \sin^2\theta + i\left(-(\cos\theta - 1)\sin\theta + (\cos\theta + 1)\sin\theta\right)}{2(1 + \cos\theta)}$$

$$= \frac{\cos^2\theta + 1 - \sin^2\theta + i\left(-\sin\theta\cos\theta + \sin\theta + \sin\theta\cos\theta + \sin\theta\right)}{2(1 + \cos\theta)}$$

$$= \frac{i\sin\theta}{1 + \cos\theta}$$

$$\tan(\theta/2) = -i\,z$$

$$\tan(\theta/2) = \frac{\sin\theta}{1 + \cos\theta}$$

$$\tan(\theta/2) = \frac{\sin\theta}{1 + \cos\theta} = \left(\frac{\sin\theta}{1 + \cos\theta}\right)\left(\frac{1 - \cos\theta}{1 - \cos\theta}\right)$$

$$= \frac{\sin\theta\,(1 - \cos\theta)}{1 - \cos^2\theta} = \frac{\sin\theta\,(1 - \cos\theta)}{\sin^2\theta}$$

$$= \frac{(1 - \cos\theta)}{\sin\theta}$$

$$\tan^2(\theta/2) = \frac{(1-\cos\theta)^2}{\sin^2\theta}$$

$$\frac{\sin^2(\theta/2)}{\cos^2(\theta/2)} = \frac{(1-\cos\theta)^2}{\sin^2\theta}$$

$$\sin^2(\theta/2) = \left[\frac{(1-\cos\theta)^2}{\sin^2\theta}\right]\cos^2(\theta/2) = \left[\frac{(1-\cos\theta)^2}{\sin^2\theta}\right]\left(1-\sin^2(\theta/2)\right)$$

$$\sin^2(\theta/2)\left[1+\frac{(1-\cos\theta)^2}{\sin^2\theta}\right] = \left[\frac{(1-\cos\theta)^2}{\sin^2\theta}\right]$$

$$\sin^2(\theta/2)\left(\sin^2\theta+1-2\cos\theta+\cos^2\theta\right) = (1-\cos\theta)^2$$

$$\sin^2(\theta/2) = \frac{(1-\cos\theta)^2}{2(1-\cos\theta)} = \frac{(1-\cos\theta)}{2} = 1-\cos^2(\theta/2)$$

$$\sin(\theta/2) = \pm\sqrt{\frac{(1-\cos\theta)}{2}}$$

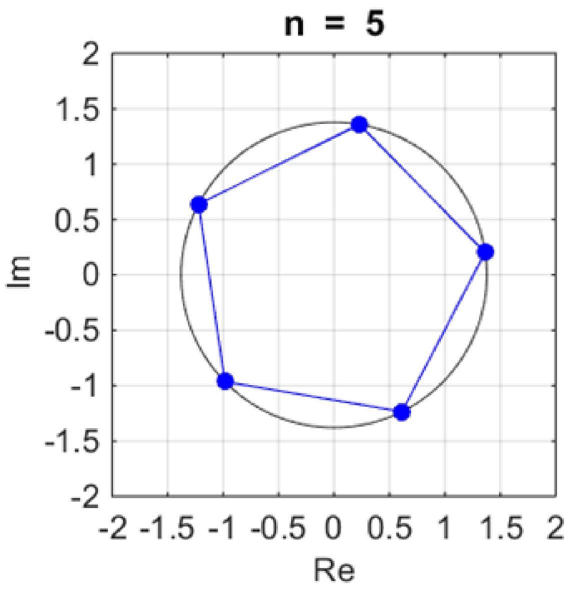
$$\cos(\theta/2) = \pm\sqrt{\frac{(1+\cos\theta)}{2}}$$

A21

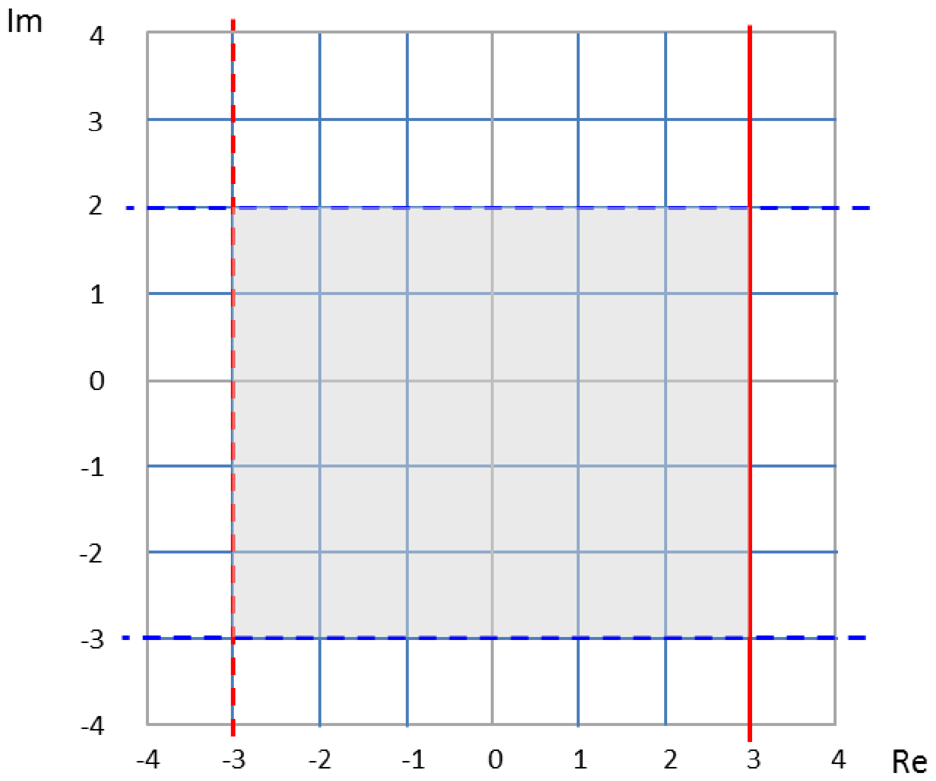
There are **five** roots for z.

$$z^5 - 1 = 0$$
$$z^5 = 1 = \cos(2\pi k) + i\sin(2\pi k) \qquad k = 0,1,2,3,4$$
$$z_k = \cos(2\pi k/5) + i\sin(2\pi k/5)$$
$$z_0 = 1$$
$$z_1 = \cos(2\pi/5) + i\sin(2\pi/5)$$
$$z_2 = \cos(4\pi/5) + i\sin(4\pi/5)$$
$$z_3 = \cos(6\pi/5) + i\sin(6\pi/5)$$
$$z_4 = \cos(8\pi/5) + i\sin(8\pi/5)$$

When these five complex numbers are plotted on an Argand diagram, they will lie on the circle $x^2 + y^2 = 1$ and be equally spaced with angular separation equal to $2\pi/5$ rad = 72^{o}.



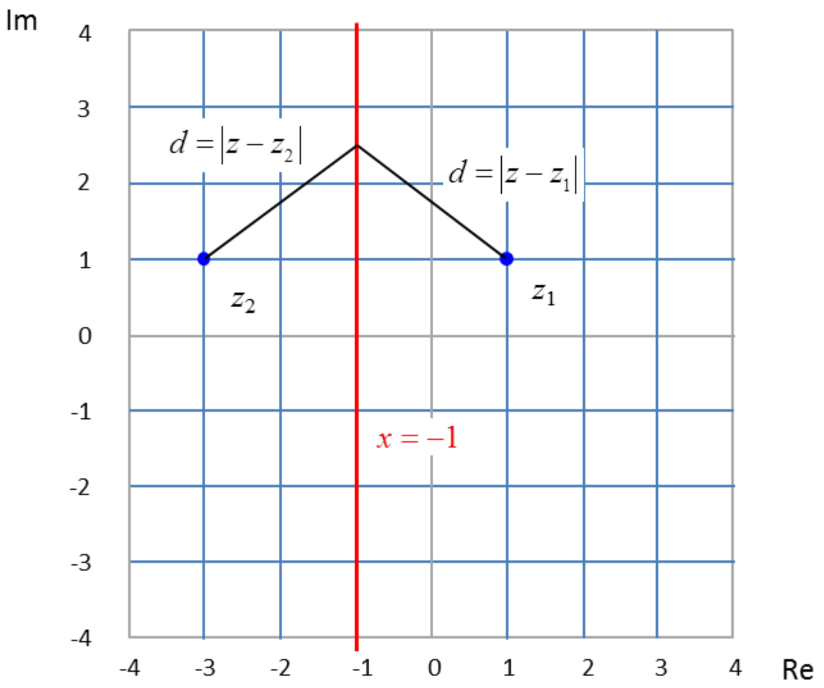
A22



A23

$$\left|(x-1)+i\,(y-1)\right|=\left|(x-1)+i\,(y-1)\right|$$
$$(x-1)^2+(y-1)^2=(x+3)^2+(y-1)^2$$
$$x^2-2x+1=x_2+6x+9$$
$$x=-1$$

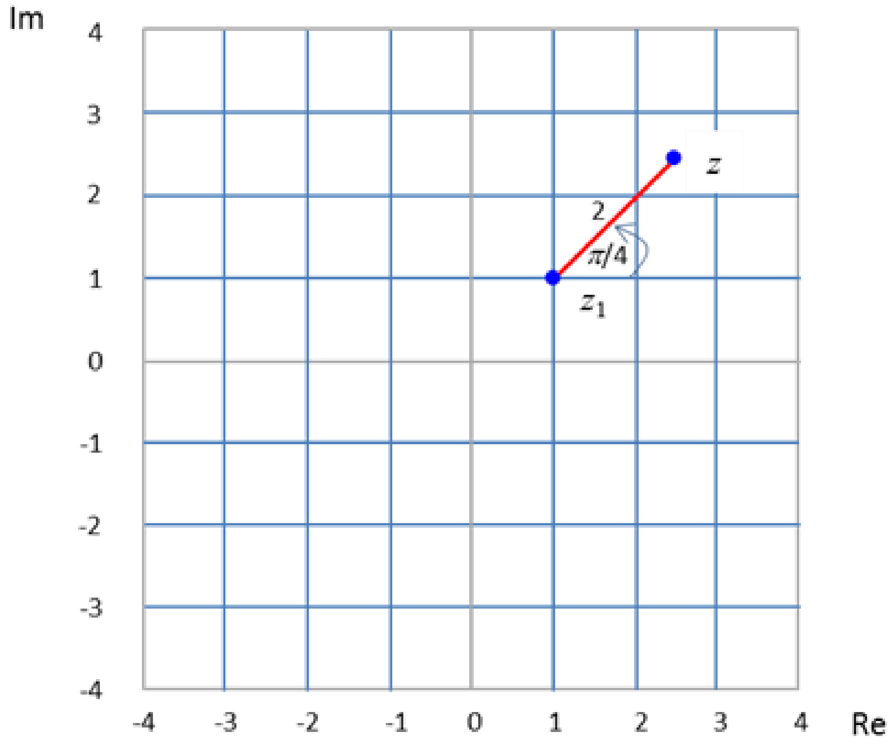
The line x = -1 corresponds to the perpendicular bisector of the two points z_1 and z_2.



A24

$$z - z_1 = 2\left(\cos\left(\pi/4\right) + i\sin\left(\pi/4\right)\right)$$
$$\left|z - z_1\right| = 2 \qquad \theta = Arg\left(z - z_1\right) = \pi/4$$

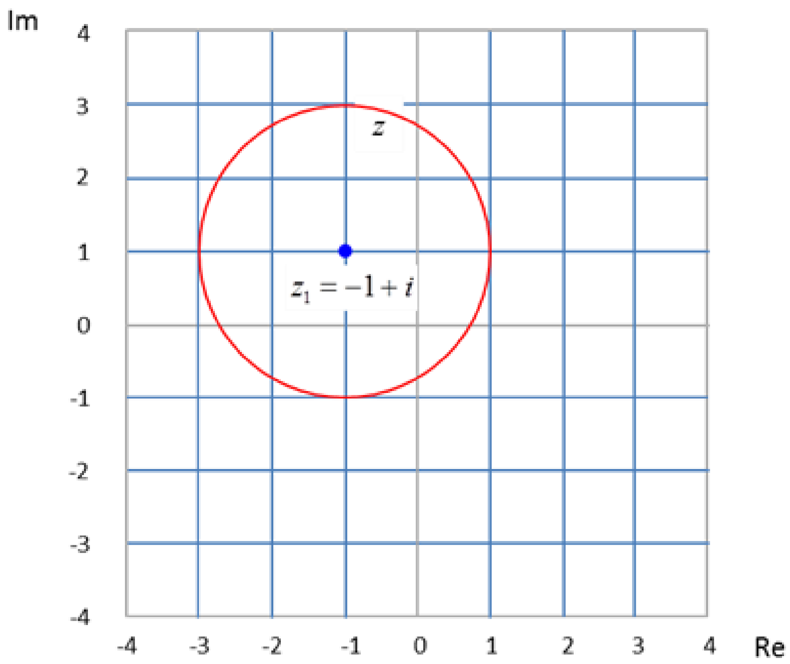
Therefore all the points z lie on the straight line drawn from (1, 1) of length 2 and at an angle of π/4 with respect to the horizontal.



A25

$$z = x + i\ y$$
$$z_1 = -1 + i$$
$$z - z_1 = (x+1) + i\,(y-1)$$
$$|z - z_1| = (x+1)^2 + (y-1)^2 = 2$$

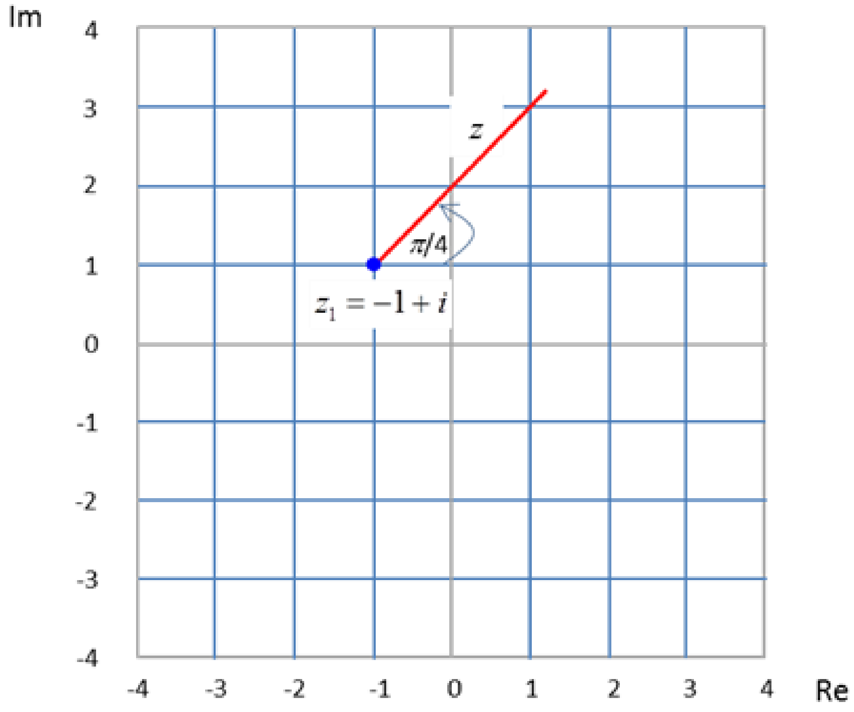
This corresponds to a circle with centre (-1,1) and radius 2.



A26

$$z = x + i\ y$$
$$z_1 = -1 + i$$
$$z - z_1 = (x+1) + i\,(y-1)$$
$$|z - z_1| = (x+1)^2 + (y-1)^2 = 2$$
$$z = x + i\ y \quad z_1 = -1 + i$$
$$z - z_1 = (x+1) + i\,(y-1)$$
$$\theta = Arg\,(z - z_1) = \text{atan}\left(\frac{y-1}{x+1}\right) = \pi/4$$
$$\tan\theta = \left(\frac{y-1}{x+1}\right)$$

The locus is the straight line from the point $z_1(-1, 1)$ but not including the point z_1 to the points z which makes an angle of $\pi/4$ with respect to the horizontal.



A27

$$z_1 = R_1 e^{i\theta_1} \qquad z_2 = R_2 e^{i\theta_2}$$

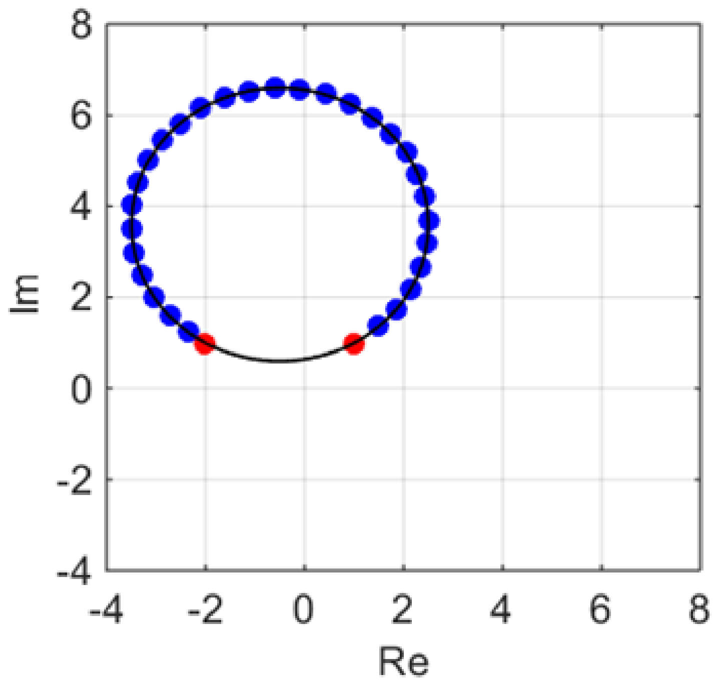
$$|z_1| = R_1 \qquad |z_2| = R_2$$

$$Arg(z_1) = \theta_1 \quad Arg(z_2) = \theta_2$$

$$\frac{z_1}{z_2} = \frac{R_1}{R_2} e^{i(\theta_1 - \theta_2)}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{R_1}{R_2} = \frac{|z_1|}{|z_2|}$$

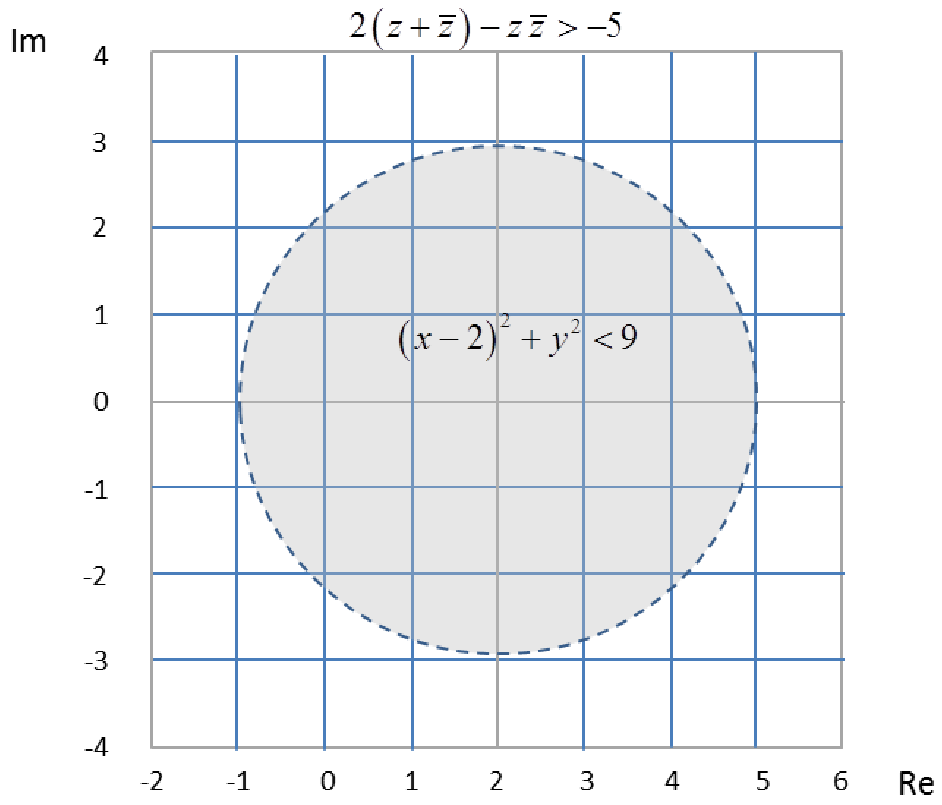
$$Arg\left(\frac{z_1}{z_2}\right) = (\theta_1 - \theta_2) = Arg(z_1) - Arg(z_2)$$



A28

$$2(z+\bar{z})-z\bar{z}>-5$$
$$z=x+i\,y \quad \bar{z}=x-i\,y$$
$$z+\bar{z}=2x \quad z\bar{z}=x^2+y^2$$
$$-(x^2+y^2)+2x>-5$$
$$x^2+y^2-4x<5$$
$$x^2-4x+4+y^2<5+4$$
$$(x-2)^2+y^2<9$$

Therefore, the allowed region is inside the circle with centre (2, 0) and radius 3. The circle is shown with a dotted line to show that the region does not include the circumference of the circle.



A29

$|z-i|<3 \quad x^2+(y-i)^2<3^2$ region inside a circle of centre (0, 1) and radius 3

$-\frac{\pi}{4}<Arg(z)<\frac{\pi}{4}$

z must lie between the lines drawn from (0, 0) and making angles of - π /4 and + π /4 with respect to the real axis.